

D=11 massless superparticle covariant quantization, pure spinor  
BRST charge and hidden symmetries

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UA61108, Kharkov, Ukraine***Abstract**

We consider the covariant quantization of the D=11 massless superparticle (M0-brane) in the spinor moving frame or twistor-like Lorentz harmonics formulation. The action involves the set of 16 constrained 32 component Majorana spinors, the spinor Lorentz harmonics  $v_{\alpha q}^-$  parametrizing (as homogeneous coordinates, modulo gauge symmetries) the celestial sphere  $S^9$ . Their presence allows us to separate covariantly the first and the second class constraints of the model. We show that, after taking into account the second class constraints by means of Dirac brackets and after further reducing the first class constraints algebra, the system is described in terms of a simple BRST charge  $\mathbb{Q}^{susy}$  associated to the  $d = 1$ ,  $n = 16$  supersymmetry algebra. The study of the cohomology of this BRST operator requires a regularization by complexifying the bosonic ghosts for the  $\kappa$ -symmetry,  $\lambda_q$ , and further reduction of the regularized cohomology problem to the one for a simpler complex BRST charge  $\tilde{\mathbb{Q}}^{susy}$ . This latter is essentially the pure spinor BRST charge by Berkovits, but with a composite pure spinor constructed from the complex  $d = 9$  spinor with zero norm,  $\tilde{\lambda}_q$ , and the spinorial harmonics  $v_{\alpha q}^-$ . This exhibits a possible origin of the complexity (non-hermiticity) characteristic of the Berkovits pure spinor approach.

The simple structure of the nontrivial cohomology of the M0-brane BRST charge  $\mathbb{Q}^{susy}$  finds explanation in the properties that the superparticle action exhibits in the so-called ‘covariantized light-cone’ basis, where the M0-brane action is expressed in terms of  $\kappa$ -symmetry invariant variables. The set of gauge symmetries in this basis reduces to the  $[SO(1,1) \times SO(9)] \ltimes K_9$  Borel subgroup of  $SO(1,10)$ . Imposing their generators as conditions on the superparticle wavefunctions, we arrive at the covariant quantization in terms of physical degrees of freedom which hints possible hidden symmetries of  $D = 11$  supergravity. Besides  $SO(16)$ , which in the twistor like Lorentz harmonic formulation is seen already at the classical level, we discuss also some indirect arguments in favor of the possible  $E_8$  symmetry.

*Keywords:* Supersymmetry, superparticle, covariant quantization, BRST, Lorentz harmonics, twistors, supergravity

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# 1 Introduction and summary

## 1.1 Introduction

A covariant quantization of the massless  $D=11$  superparticle (see [1, 2]) has been recently considered [3] in its twistor-like Lorentz harmonics formulation [4] (see also [5, 6, 7, 8]). This new, covariant *supertwistor quantization* leads to the linearized  $D=11$  supergravity multiplet in the spectrum (in agreement with the light-cone results of [2]) and permitted to find a possible origin of the hidden  $SO(16)$  symmetry of the  $D = 11$  supergravity [9]. In this paper we study the BRST quantization of the  $D=11$  massless superparticle model in that approach and then turn back to the covariant quantization of physical degrees of freedom (different from the supertwistor one in [3]) to search for an explanation of the simple structure of the superparticle cohomologies.

The  $D = 11$  superparticle is interesting on its own, as the simplest of the M-theory superbranes, the M0-brane, and because its quantization produces, as noticed above, the linearized  $D=11$  supergravity multiplet. Nevertheless, our main motivation is to look for the origin and geometric meaning of the ‘pure spinor’ formalism by Berkovits [10]. Recently, a breakthrough in the covariant description of quantum superstring theory has been reached in this pure spinor framework: a technique for loop calculations was developed [11] and the first results were given in [11, 12, 13]. In particular, two new multiloop theorems useful in a recent investigations of the possible finiteness of  $N=8$   $D=4$  supergravity [14] were proved in [13]. On the other hand, the pure spinor superstring was introduced -and still remains- as a set of prescriptions for quantum superstring calculations, rather than as a quantization of the Green-Schwarz superstring. Despite a certain progress in relating the pure spinor superstring [10] to the original Green-Schwarz formulation [15], and also [16] to the superembedding approach [17, 18, 19, 20]<sup>1</sup>, the origin and geometrical meaning of the pure spinor formalism is far from being clear. Possible modifications of the pure spinor approach are also being considered (see *e.g.* [26]).

In this context, the Lorentz harmonic approach [28, 29, 30, 5, 31, 6, 7, 8, 32], in the frame of which a significant progress in solving the problem of covariant superstring quantization had already been made in late eighties [29, 30], looks particularly interesting. Although no counterpart of the recent progress in loop calculations [11, 12] has been reached (yet) in the Lorentz harmonics framework, its relation with the superembedding approach [17, 18, 19, 20], transparent geometrical meaning [28, 31, 6, 7, 8] and twistor-likeness [6, 7, 8] justifies the hope that its further development (in the pragmatic spirit characteristic for the pure spinor approach of [10, 11, 12]) may be helpful to understand the origin and the geometrical meaning of the pure spinor formalism [10] as well as its nonminimal modifications [26] and even that it might provide a basis for an alternative, convenient and algorithmic, technique for the superstring loop calculations. A natural first stage in such a program is to study the covariant quantization of superparticle, and in particular, of the  $D=11$  massless superparticle or M0-brane<sup>2</sup>, less studied as well in comparison with the  $D=10$  and  $D=4$  superparticle models.

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<sup>1</sup>Notice also the recent progress [21] in derivation of the pure spinor ghost measure for loop calculations, which was originally proposed in [11] on the ground of a series of very elegant but indirect arguments involving the picture changing operator characteristic of the RNS (Ramond-Neveu-Schwarz) string model [22, 23]. This was reached, however, by starting from the pure spinor superstring by Berkovits, covariantizing it with respect to the worldsheet reparametrizations by introducing two dimensional gravity and quantizing this sector *à la* Batalin-Vilkovisky [24]. Thus, although the subject of [21] was the quantization of Berkovits pure spinor model rather than the original Green-Schwarz superstring, a deeper understanding of the loop calculation technique has been reached already at this stage. The approach similar to [21] was also developed in earlier [25].

<sup>2</sup>See [27] for the loop calculations with the use of the  $D = 11$  pure spinor formalism.

## 1.2 Summary of the main results

The BRST charge proposed by Berkovits [10] has the form

$$\mathbb{Q}^B = \Lambda^\alpha d_\alpha , \quad (1.1)$$

where  $d_\alpha$  are the fermionic constraints of the (here  $D=11$ ) superparticle model, which obey the algebra

$$\{d_\alpha, d_\beta\} = 2iP_{\alpha\beta} \equiv 2i\Gamma_{\alpha\beta}^m P_m \quad (\text{here } \alpha = 1, \dots, 32, m = 0, 1, \dots, 9, \#), \quad (1.2)$$

where  $P_m$  is the superparticle momentum, and  $\Lambda^\alpha$  is the complex *pure spinor* which obeys

$$\Lambda \Gamma_a \Lambda = 0, \quad \Lambda^\alpha \neq (\Lambda^\alpha)^* . \quad (1.3)$$

This constraint guarantees the nilpotency  $(\mathbb{Q}^B)^2 = 0$  of the Berkovits BRST charge (1.1).

The generic null spinor  $\Lambda_\alpha$  contains 23 complex or 46 real parameters [10]<sup>3</sup>. A 39 parametric solution  $\tilde{\Lambda}_\alpha$  of this constraint is provided by

$$\tilde{\Lambda}_\alpha = \tilde{\lambda}_p^+ v_{\alpha p}^-, \quad \tilde{\lambda}_p^+ \tilde{\lambda}_p^+ = 0, \quad \{v_{\alpha p}^-\} = \frac{Spin(1, 10)}{[Spin(1, 1) \otimes Spin(9)] \ltimes \mathbb{K}_9} = \mathbb{S}^9, \quad (1.4)$$

where  $\tilde{\lambda}_p^+$  is a complex 16 component  $SO(9)$  spinor with zero norm,  $\tilde{\lambda}_p^+ \tilde{\lambda}_p^+ = 0$ , carrying  $32 - 2 = 30$  degrees of freedom and  $v_{\alpha p}^-$  are spinorial Lorentz harmonics [4] (see also [5, 31], [6, 7, 8, 4]), a set of 16 constrained  $D=11$  bosonic spinors which, once the constraints are taken into account, provide the homogeneous coordinates for the 11 dimensional celestial sphere  $S^9$  and thus carry 9 degrees of freedom (see below). The existence of such a solution already suggests a relation among the pure spinor and the Lorentz harmonics approaches.

Notice that in  $D = 10$  dimensional case such a relation is much more close. The solution of Eq. (1.4) carries **16+8-2=22** degrees of freedom, the same number as the generic pure spinor, so that it is the general solution. This may be important for the study of superstring covariant quantization on the line similar to what we present here for the case of superparticle.

Here we first construct the Hamiltonian mechanics of the twistor-like Lorentz harmonics formulation of the  $D = 11$  superparticle and, with the help of the spinorial Lorentz harmonics, separate *covariantly* the first and the second class constraints (see [7] for an analogous result for the Green-Schwarz superstring). Then we take into account the second class constraints by introducing Dirac brackets [40], and calculate the Dirac brackets algebra of the first class constraints which happens to be a nonlinear algebra. Further, following the pragmatic spirit of the Berkovits' approach [10], [11], we take care of the part of constraints separately and left with a set of 16 fermionic and 1 bosonic first class constraints, the generators of the fermionic  $\kappa$ -symmetry (see [36, 37]) and its bosonic  $b$ -symmetry superpartner, the Dirac brackets of which represent the  $d = 1$ ,  $n = 16$  (worldline) supersymmetry algebra. This set of constraints is described by the BRST charge

$$\mathbb{Q}^{susy} = \lambda_q^+ D_q^- + i c^{++} \partial_{++} - \lambda_q^+ \lambda_q^+ \frac{\partial}{\partial c^{++}}, \quad \{D_p^-, D_q^-\} = 2i\delta_{qp} \partial_{++}, \quad (1.5)$$

including 16 real bosonic ghosts  $\lambda_q^+$  and <sup>4</sup> one real fermionic ghost  $c^{++}$ .

<sup>3</sup>The direct counting shows  $32 - 11 = 21$  complex or 42 real parameters, but one can show, passing to the  $SO(1, 9)$  covariant representation of the (originally  $SO(1, 10)$  covariant)  $D=11$  pure spinor condition [10], that two of the 11 complex conditions are satisfied automatically, so that there are only nine independent complex conditions.

<sup>4</sup>The sign superscripts in  $\lambda_q^+$  and  $D_q^-$  denote the spinorial Majorana-Weyl (MW) representations of  $SO(1, 1)$ ; double sign superscript  $--$ ,  $++$  or subscript, like in  $\partial_{++}$ , would correspond to the  $SO(1, 1)$  vector. Since the MW spinorial representation of  $SO(1, 1)$  is one dimensional, the subscript  $+$  is equivalent to superscript  $-$  and *vice versa*, so that  $\{D_p^-, D_q^-\} = 2i\delta_{qp} \partial_{++}$  in (1.5) is  $SO(1, 1)$  invariant. This notation corresponds to the light-cone basis in two dimensional space (or in two dimensional subspace of the  $D$ -dimensional spacetime) with the (flat space) metric of the form  $g_{++} g_{--} = \frac{1}{2}$ ,  $g_{++} g_{++} = 0 = g_{--} g_{--}$ , so that, *e.g.*  $\partial_{++} = \frac{1}{2} \partial^{--}$ , where the coefficients  $\frac{1}{2}$  (which then appear in Eq. (2.19)) are introduced to avoid the appearance of  $\sqrt{2}$  coefficients in many equations.

An analysis of the cohomology of this BRST operator shows that it is trivial if the norm  $\lambda_q^+ \lambda_q^+$  of bosonic ghost  $\lambda_q^+$  is nonvanishing. In other words, the nontrivial cohomology of  $\mathbb{Q}^{susy}$  has support on  $\lambda_q^+ \lambda_q^+ = 0$ . For a real spinor  $\lambda_q^+ \lambda_q^+ = 0$  implies  $\lambda_q^+ = 0$ . This produces a technical problem which is sorted out by means of a regularization which consists in allowing  $\lambda_q^+$  to be *complex*,  $\lambda_q^+ \mapsto \tilde{\lambda}_q^+ \neq (\tilde{\lambda}_q^+)^*$  so that  $\tilde{\lambda}_q^+ \tilde{\lambda}_q^+ = 0$  allows for nonvanishing complex solutions. Furthermore, this implies the reduction of the cohomology problem for the regularized BRST operator  $\mathbb{Q}^{susy}$  to the search for cohomology at vanishing bosonic ghost,  $\tilde{\lambda}_q^+ = 0$ , for the following complex BRST charge

$$\tilde{\mathbb{Q}}^{susy} = \tilde{\lambda}_q^+ D_q^- + i c^{++} \partial_{++}, \quad \tilde{\lambda}_q^+ \tilde{\lambda}_q^+ = 0, \quad \{D_p^-, D_q^-\} = 2i \delta_{qp} \partial_{++}. \quad (1.6)$$

We discuss the relation of the above non-hermitean  $\tilde{\mathbb{Q}}^{susy}$  operators with the (always complex) Berkovits BRST charge and find that this comparison shows the possible origin of the intrinsic complexity of the Berkovits formalism.

The above results were briefly reported in [41]; here we give details on their derivation. The possible results of stringy generalizations are discussed in the concluding sec. 6 of the present paper.

Let us stress that of all the cohomologies of the Berkovits-like BRST charge  $\tilde{\mathbb{Q}}^{susy}$  (1.6) only the ones calculated (and remaining nontrivial) at  $\tilde{\lambda}_q = 0$  describe the cohomology of the superparticle BRST operator  $\mathbb{Q}^{susy}$ . The full cohomology of  $\tilde{\mathbb{Q}}^{susy}$  is clearly richer and is related with spinorial cohomologies of [42]. As far as the  $\tilde{\mathbb{Q}}^{susy}$  cohomology for vanishing bosonic ghost describing the M0-brane spectrum is concerned, this can be described by the function of variables which are inert under  $\kappa$ - and  $b$ -symmetry. This relatively simple structure finds its explanation in the properties of the superparticle action in the so-called covariantized light-cone basis (see [28, 5, 32]).

The change of variable corresponding to this basis in the superparticle spinor moving frame action results in an automatical gauge fixing of the  $\kappa$ -symmetry and  $b$ -symmetry. Thus, in this basis, the set of superparticle first class constraints contains only the generators of the Borel subgroup  $[SO(1,1) \times SO(9)] \ltimes K_9$  of the Lorentz group  $SO(1,10)$ . We present here the BRST charge describing this set of the first class constraints. Then, following Dirac [40], we impose this set of first class constraints as conditions on the wave function and discuss the quantization of physical degrees of freedom (a covariantized light cone basis prototype of the supertwistor quantization in [3]) which shows the hints of hidden  $SO(16)$  symmetry and suggests some speculations on possible  $E_8$  symmetry of  $D = 11$  supergravity.

### 1.3 Structure of the paper

This paper is organized as follows. Sec. 2 reviews the spinor moving frame (twistor like Lorentz harmonics) formulation of the  $D=11$  massless superparticle or M0-brane and shows its classical equivalence with the standard Brink-Schwarz formulation. In Sec. III we develop the Hamiltonian formalism for this formulation of M0-brane, discuss its classical BRST charge and the reduced BRST operator  $\mathbb{Q}^{susy}$  corresponding to a subset of the M0-brane first class constraints. Particularly, the primary constraints are obtained in sec. 3.1. In sec. 3.2 the Dirac brackets that allow us to treat harmonic variables as coordinates on the Lorentz group manifold are defined. These are related with the group-theoretical structure of Lorentz harmonics in sec. 3.3, where the  $SO(1,10)/[SO(1,1) \otimes SO(9)] \ltimes K_9$  Cartan forms are introduced. These are used in sec. 3.4 to define canonical Hamiltonian of the M0-brane model. The second class constraints are found and the Dirac brackets allowing to treat them in the strong sense are presented in sec. 3.5. The Dirac bracket algebra of all the first class constraints is presented in sec. 3.6. The BRST charge  $\mathbb{Q}'$  for the nonlinear (sub)superalgebra of the first class constraints is obtained in sec. 3.7. Finally, the BRST charge  $\mathbb{Q}^{susy}$  is obtained by reduction of  $\mathbb{Q}'$  in sec. 3.8.

The cohomology of  $\mathbb{Q}^{susy}$  is studied in Sec. 4. In particular, the complex charge (1.6) is introduced in sec. 4.2 and its relation with the Berkovits BRST charge is discussed in sec. 4.3. To

explain the relatively simple structure of the  $\mathbb{Q}^{susy}$  cohomology, in Sec. 5 we study the superparticle spinor moving frame action in the covariantized light-cone basis (see [28, 5, 32]). The automatical gauge fixing of the  $\kappa$ -symmetry and  $b$ -symmetry which occurs in the action when changing variables to this basis is discussed in sec. 5.1.1. The BRST charge describing the set of the first class constraints of the superparticle action in this basis is presented in sec. 5.1.2. The quantization of the physical degrees of freedom of the superparticle using the covariantized light cone basis is discussed in sec. 5.2. There we also discuss the hints of possible hidden symmetries of the D=11 supergravity which appears on the way of such a covariant quantization.

In Sec. 6 we present our conclusions (sec. 6.1) and an outlook (secs. 6.2, 6.3), including the discussion on possible results of the generalization of our study of M0-brane to the case of type IIB superstring (sec. 6.2). Some technical details on harmonics are presented in the Appendix.

## 2 The M0-brane in the spinor moving frame formulation. Twistor-like action and its gauge symmetries.

### 2.1 Towards the spinor moving frame action for the D=11 massless superparticle

The Brink-Schwarz massless superparticle action,  $S_{BS} = \int_{W^1} \frac{1}{2e} \Pi_{\tau m} \Pi_{\tau}^m$ , can be written in the following first order form

$$S_{BS}^1 = \int_{W^1} \left( P_m \Pi^m - \frac{1}{2} d\tau e P_m P^m \right), \quad (2.1)$$

where  $P_m(\tau)$  is the auxiliary momentum variable,  $e(\tau)$  is the worldline einbein and  $\Pi^m = d\tau \hat{\Pi}_{\tau}^m$  is the pull-back of the bosonic supervielbein of the tangent superspace to the superparticle worldline. In flat  $D = 11$  superspace this reads

$$\Pi^m := dx^m - id\theta \Gamma^m \theta = d\tau \hat{\Pi}_{\tau}^m, \quad \hat{\Pi}_{\tau}^m := \partial_{\tau} \hat{x}^m(\tau) - i \partial_{\tau} \hat{\theta}(\tau) \Gamma^m \hat{\theta}(\tau) \quad (2.2)$$

The action (2.1) is valid in any dimension; the  $D=11$  massless superparticle action [1] corresponds to  $m = 0, 1, \dots, 9, \#$ , a 32 component Majorana spinor  $\theta^{\alpha}$  and  $32 \times 32$  eleven-dimensional gamma matrices  $\Gamma_{\alpha\beta}^m := \Gamma_{\alpha}^m \gamma C_{\gamma\beta} = \Gamma_{\beta\alpha}^m$ .

The einbein  $e(\tau)$  plays the rôle of Lagrange multiplier and produces the mass shell constraint

$$P_m P^m = 0. \quad (2.3)$$

Since Eq. (2.3) is algebraic, it may be substituted into the action (2.1), which gives

$$S'_{M0} = \int_{W^1} P_m \hat{\Pi}^m, \quad P_m P^m = 0. \quad (2.4)$$

Thus, if the general solution of (2.3) is known, one may substitute it for  $P_m$  in (2.4) and obtain a classically equivalent formulation of the  $D$ - (here 11-) dimensional Brink-Schwarz superparticle. The moving frame or twistor-like Lorentz harmonics formulation of [4, 3] (see [5] for  $D=4$  and [34] for  $D=10$ ) can be obtained just in this way.

It is easy to solve the constraint (2.3) in a non-covariant manner: in a special Lorentz frame a solution with positive energy,  $\hat{P}_{(a)}$ , reads *e.g.*

$$\hat{P}_{(a)} = \frac{\rho}{2} (1, \dots, -1) = \frac{\rho}{2} (\delta_{(a)}^0 - \delta_{(a)}^{\#}) \quad . \quad (2.5)$$

The solution in an arbitrary frame follows from (2.5) by making a Lorentz transformation,  $P_m = U_m^{(a)} \hat{P}_{(a)}$  with  $U_m^{(a)} \in SO(1, 10)$ ,

$$P_m := U_m^{(a)} \hat{P}_{(a)} = \frac{\rho}{2} (u_m^0 - u_m^\#), \quad U_m^{(a)} := (u_m^0, u_m^i, u_m^\#) \in SO(1, D-1). \quad (2.6)$$

Note that, since  $P_m = P_m(\tau)$  is dynamical variable in the action (2.4), the same is true for the Lorentz group matrix  $U$  when it is used to express  $P_m$  through Eq. (2.6),  $U_m^{(a)} = U_m^{(a)}(\tau) = (u_m^0(\tau), u_m^i(\tau), u_m^\#(\tau))$ . Such *moving frame variables* [6] are called *Lorentz harmonics* [5, 31] (see [38], also light-cone harmonics in [28]).

Substituting (2.6) for  $P_m$  in (2.4) or, equivalently, in (2.1), one arrives at the following action

$$S = \int_{W^1} \frac{1}{2} \rho^{++} u_m^{--} \hat{\Pi}^m, \quad u_m^{--} u^{--m} = 0 \quad (\Leftarrow \quad U := \{\frac{u_m^{++} + u_m^{--}}{2}, u_m^i, \frac{u_m^{++} - u_m^{--}}{2}\} \in SO(1, 10)) \quad (2.7)$$

where the light-likeness of the vector  $u_m^{--} = u_m^0 - u_m^\#$  (see also (2.18) below) follows from the orthogonality and normalization of the timelike  $u_m^0$  and spacelike  $u_m^\#$  vectors which, in their turn, follow from  $U \in SO(1, 10)$  in Eq. (2.6) (as it is noticed in the brackets in (2.7)).

At this stage it might seem obscure what is the advantage of the action of Eq. (2.7) with respect to (2.4) or (2.1). However, as we discussed below, the action (2.7) *hides the twistor-like action*, a higher dimensional ( $D=11$  here) generalization of the  $D=4$  Ferber–Schirafuji action [35]. The twistor like variables called *spinorial harmonics* appears as ‘square roots’ of the vector harmonics (see below); they can be used to separate covariantly the first and the second class constraints and to provide the *irreducible* form of the  $\kappa$ -symmetry [36, 37] (infinitely reducible in the standard formulation of massless superparticle [37]<sup>5</sup>). This also explains why the formulation based on the action (2.7) is called the *spinor moving frame* formulation.

## 2.2 Twistor-like spinor moving frame action of M0-brane and its gauge symmetries.

The spinor moving frame action for the  $D = 11$  massless superparticle can be written in the following equivalent forms [4] (see [34] for  $D=10$  and [5] for  $D=4$ )

$$S := \int d\tau L = \int_{W^1} \frac{1}{2} \rho^{++} u_m^{--} \Pi^m = \int_{W^1} \frac{1}{32} \rho^{++} v_{\alpha q}^- v_{\beta q}^- \Pi^m \tilde{\Gamma}_m^{\alpha\beta}, \quad (2.8)$$

$$\alpha = 1, 2, \dots, 32 \quad (n \text{ in general}), \quad q = 1, \dots, 16 \quad (n/2 \text{ in general}),$$

$$m = 0, \dots, 9, \# \quad ((D-1) \text{ in general})$$

where we use the symbol  $\#$  to denote the tenth spatial direction ( $X^\# := X^{10}$ ) and the notation  $\Gamma^m \equiv \Gamma^m_{\alpha\beta} := \Gamma^m_{\alpha}{}^{\gamma} C_{\gamma\beta}$ ,  $\tilde{\Gamma}^m \equiv \tilde{\Gamma}^m{}^{\alpha\beta} := C^{\alpha\gamma} \Gamma^m_{\gamma}{}^{\beta}$  for the  $D = 11$  gamma-matrices contracted with  $C_{\alpha\beta}$  and  $C^{\alpha\beta}$ . The first from of the action (2.8) coincides with (2.7); the second form is twistor-like, *i.e.* it resembles the Ferber–Schirafuji action [35] for the massless  $D = 4$  superparticle. Instead of two-component Weyl spinor of the Ferber supertwistor, the action of Eq. (2.8) includes the set of 16 bosonic 32-component Majorana spinors  $v_{\alpha q}^-$  which satisfy the following kinematical constraints (see [6, 7, 4]),

$$\begin{cases} 2v_{\alpha q}^- v_{\beta q}^- &= u_m^{--} \Gamma_{\alpha\beta}^m & (a), \\ v_q^- \tilde{\Gamma}_m v_p^- &= \delta_{qp} u_m^{--} & (b), \\ v_{\alpha q}^- C^{\alpha\beta} v_{\beta p}^- &= 0 & (c), \end{cases} \quad u_m^{--} u^{m--} = 0 \quad (d). \quad (2.9)$$

<sup>5</sup>Notice that in the case of massless  $N=2$  superparticle, which presently are identified with D0-branes, the covariant gauge fixing of the  $\kappa$ -symmetry is possible already in the standard formulation [36].

Although, in principle, one can study the dynamical system using just the kinematical constraints (2.9) (see [39, 44]), it is more convenient to treat the light-like vector  $u_m^-$  as an element of the  $SO(1,10)$ -valued matrix describing *vector moving frame* and the set of 16  $SO(1,10)$ - spinors  $v_{\alpha q}^-$  as part of the corresponding  $Spin(1,10)$ -valued matrix describing the *spinor moving frame*. These moving frame variables are also called (*vector* and *spinor*) *Lorentz harmonics* and will be discussed in Sec. 2.4 below.

Let us conclude this section by noticing that the action (2.8) possesses a set of gauge symmetries which includes

i) the *irreducible*  $\kappa$ -symmetry

$$\delta_\kappa x^m = i\delta_\kappa \theta^\alpha \Gamma_{\alpha\beta}^m \theta^\beta, \quad \delta_\kappa \theta^\alpha = \kappa^{+q} v_q^{-\alpha}, \quad \delta_\kappa v_{\alpha q}^- = 0 = \delta_\kappa u_m^-; \quad (2.10)$$

the possibility to reformulate the  $\kappa$ -symmetry in the irreducible form is due to the presence of the constrained bosonic spinor variables  $v_{\alpha q}^-$  (see [6, 34] and the discussion below);

ii) its superpartner, the tangent space copy of the worldvolume reparametrization symmetry, which we, following the pioneer paper [36], call *b*-symmetry,

$$\delta_b x^m = b^{++} u_m^{-m}, \quad \delta_b \theta^\alpha = 0, \quad \delta_b v_{\alpha q}^- = 0 = \delta_b u_m^-; \quad (2.11)$$

iii) a scaling  $GL(1, \mathbb{R})$  symmetry

$$\rho^{++} \mapsto e^{2\alpha} \rho^{++}, \quad u_m^- \mapsto e^{-2\alpha} u_m^-, \quad v_{\alpha q}^- \mapsto e^{-\alpha} v_{\alpha q}^-, \quad (2.12)$$

with the wait determined by the sign indices  $^{++}$ ,  $^{--}$  and  $^-$ . In the light of Lorentz harmonic treatment of  $v_{\alpha q}^-$  and  $u_m^-$ , which will be presented below, we prefer to identify this scaling symmetry as  $SO(1,1)$  group transformations.

iv) The action (2.8) is also invariant under the  $Spin(9)$  symmetry acting on the  $q = 1, \dots, 16$  index of the constrained bosonic spinor variable  $v_{\alpha q}^-$ ,

$$v_{\alpha q}^- \mapsto v_{\alpha p}^- S_{pq}, \quad S_{pq} \in Spin(9) \quad \Leftrightarrow \quad \begin{cases} S^T S = \mathbb{I}_{16 \times 16}, \\ S \gamma^I S^T = \gamma^J U^{JI}, \end{cases} \quad U^T U = \mathbb{I}_{9 \times 9}, \quad (2.13)$$

Notice that the nine dimensional charge conjugation matrix is symmetric and can be identified with the Kronecker delta symbol,  $\delta_{qp}$ , so that the contraction  $v_{\alpha q}^- v_{\beta q}^-$ , entering the action, is  $Spin(9)$  invariant.

This  $Spin(9)$  symmetry is used as an identification relation when the spinorial Lorentz harmonics are defined as homogeneous coordinates of the coset  $\frac{SO(1,10)}{[SO(1,1) \otimes SO(9)] \otimes K_9 = \mathbb{S}^9}$  (see below) given by a  $Spin(1,10)$  valued matrix  $V_\alpha^{(\beta)} = (v_{\alpha q}^-, v_{\alpha q}^+) \in Spin(1,10)$ , one of the two  $32 \times 16$  blocks of which is identified with our  $v_{\alpha q}^-$ .

However, when the action (2.8) with the variable  $v_{\alpha q}^-$  subject only to the constraints (2.9) is considered, one immediately finds that neither constraints nor the action involve the  $d = 9$  gamma matrices; all the contractions are made with  $16 \times 16$  Kronecker symbol  $\delta_{qp}$ , and the same matrix only is used in the constraints.

### 2.3 On $O(16)$ gauge symmetry

Thus we have observed that *the action* (2.8), when considered as constructed from spinorial variables restricted by the constraints (2.9),

$$S = \int_{W^1} \frac{1}{32} \rho^{++} \tilde{v}_{\alpha q}^- \tilde{v}_{\beta q}^- \Pi^m \tilde{\Gamma}_m^{\alpha\beta}, \quad \begin{cases} 2\tilde{v}_{\alpha q}^- \tilde{v}_{\beta q}^- = \frac{1}{16} \tilde{v}_{p'}^- \tilde{\Gamma}_m \tilde{v}_{p'}^- \Gamma_{\alpha\beta}^m, & (a) \\ \tilde{v}_q^- \tilde{\Gamma}_m \tilde{v}_{p'}^- = \delta_{qp} \frac{1}{16} \tilde{v}_{p'}^- \tilde{\Gamma}_m \tilde{v}_{p'}^-, & (b) \\ \tilde{v}_{\alpha q}^- C^{\alpha\beta} \tilde{v}_{\beta q}^- = 0, & (c) \end{cases} \quad (2.14)$$

$$\alpha = 1, 2, \dots, 32, \quad q = 1, \dots, 16,$$



actually possesses the local  $SO(16)$  symmetry acting on the  $q = 1, \dots, 16$  indices of  $\tilde{v}_{\alpha q}^-$  variables,

$$\tilde{v}_{\alpha q}^- \mapsto \tilde{v}_{\alpha p}^- O_{pq}, \quad O_{pq} \in O(16) \quad \Leftrightarrow \quad O^T O = \mathbb{I}_{16 \times 16}. \quad (2.15)$$

One can conclude that the relation between spinorial harmonic  $v_{\alpha q}^-$ , which transforms under  $Spin(9)$  symmetry, and the above  $\tilde{v}_{\alpha p}^-$ , carrying the  $SO(16)$  index  $p$  is given by

$$\tilde{v}_{\alpha p}^- = v_{\alpha q}^- L_{qp}, \quad L_{qp} \in O(16) \quad \Leftrightarrow \quad L^T L = \mathbb{I}_{16 \times 16}, \quad (2.16)$$

where  $L_{qp}$  is an arbitrary orthogonal  $16 \times 16$  matrix. Clearly,  $\tilde{v}_{\alpha p}^-$  of Eq. (2.16) solves the constraints (2.9a-d) if these are solved by  $v_{\alpha q}^-$ . But if  $v_{\alpha q}^-$  is the spinorial harmonic, this is to say a  $32 \times 16$  block of the  $Spin(1, 10)$  valued matrix  $V_\alpha^{(\beta)} = (v_{\alpha q}^-, v_{\alpha q}^+) \in Spin(1, 10)$ , then  $\tilde{v}_{\alpha p}^-$  cannot be such a block if the  $O(16)$  matrix  $L_{pq}$  does not belong to the  $Spin(9)$  subgroup of  $SO(16)$ . However,  $\tilde{v}_{\alpha p}^- \tilde{v}_{\beta p}^- = v_{\alpha q}^- v_{\beta q}^-$  so that substituting (2.16) for  $\tilde{v}_{\alpha p}^-$  in (2.14), one observes the cancelation of the contributions of the matrix  $L_{qp}$ .

On one hand this is tantamount to the statement of the  $O(16)$  of the action (2.14), with variable restricted only by the constraints presented explicitly. On the other hand, this can be used to treat the variables  $v_{\alpha q}^-$  in the action (2.8) as spinorial harmonics (allowing only the  $Spin(9)$  transformations (2.13) on  $q$  index). In the next section we accept this latter point of view as it is technically more convenient for the Hamiltonian analysis. The reason is that the constraints (2.9) are reducible<sup>6</sup> and even to calculate the number of degrees of freedom becomes a nontrivial problem. This can be solved passing through the identification of  $v_{\alpha q}^-$  with spinorial harmonics: also one introduces additional variables  $v_{\alpha q}^+$ , one gains a clear group theoretical and geometrical meaning which helps to deal with the reducible constraints.

To conclude this section, let us note that the (seemingly fictitious)  $SO(16)$  symmetry of the M0-brane, which we have observed studying different versions of its twistor-like formulation, reappears inevitably in the quantization of physical degrees of freedom which we will consider in Sec. 5 (see also [3]).

## 2.4 Vector and spinor Lorentz harmonics: moving frame and spinor moving frame

The *vector* Lorentz harmonics variables  $u_m^{\pm\pm}$ ,  $u_m^i$  [28] are defined as elements of the  $SO(1, 10)$  Lorentz group matrix, Eq. (2.6). In the lightlike basis they are given by

$$U_m^{(a)} = (u_m^{--}, u_m^{++}, u_m^i) \in SO(1, 10), \quad (2.17)$$

$$m = 0, 1, \dots, 9, \# , \quad (a) = ++, --, i, \quad i = 1, \dots, 9 ,$$

where  $u_m^{\pm\pm} = u_m^0 \pm u_m^\#$ . The three-blocks splitting (2.17) is invariant under  $SO(1, 1) \otimes SO(9)$ ;  $SO(1, 1)$  rotates  $u_m^0$  and  $u_m^\#$  among themselves and, hence, transforms their sum and differences,  $u_m^{\pm\pm} = u_m^0 \pm u_m^\#$ , by inverse scaling factors, see Eq. (2.12). The fact that  $U \in SO(1, 10)$  implies the following set of constraints

$$U^T \eta U = \eta \quad \Leftrightarrow \quad \begin{cases} u_m^{--} u^{m--} = 0, & u_m^{++} u^{m++} = 0, & u_m^{\pm\pm} u^{m i} = 0, \\ u_m^{--} u^{m++} = 2, & u_m^i u^{m j} = -\delta^{ij} \end{cases} \quad (2.18)$$

or, equivalently, the unity decomposition

$$\delta_m^n = \frac{1}{2} u_m^{++} u^{n--} + \frac{1}{2} u_m^{--} u^{n++} - u_m^i u^{n i} \quad \Leftrightarrow \quad U \eta U^T = \eta. \quad (2.19)$$

---

<sup>6</sup>This is seen already from the fact that their number, 2122, exceed the number 512 of the components of  $32 \times 16$  matrix. The above number of constraints is composed as  $2122 = 528 - 11 + 1496 - 11 + 120$ , where  $-11$  come from the facts of coincidence of the gamma-trace parts of constraints (a) and (b) and of that  $u_m^{--}$  can be defined by means of one of these parts; the light-likeness of  $u_m^{--}$ , Eq. (2.9d), follows from the fact that the rank of the matrix in the *l.h.s.* of the constraint (2.9a) is 16 or less and, thus, is not counted.

The *spinor* harmonics [31] or spinor moving frame variables [6, 7, 8]  $v_{\alpha q}^{\pm}$  are elements of the  $32 \times 32$   $Spin(1, 10)$ -valued matrix

$$V_{\alpha}^{(\beta)} = (v_{\alpha q}^{-}, v_{\alpha q}^{+}) \in Spin(1, 10) \quad (\alpha = 1, \dots, 32, q = 1, \dots, 16) . \quad (2.20)$$

They are ‘square roots’ of the associated vector harmonics in the sense that

$$V\Gamma^{(a)}V^T = \Gamma^m U_m^{(a)} \quad (a) , \quad V^T \tilde{\Gamma}_m V = U_m^{(a)} \tilde{\Gamma}_{(a)} \quad (b) , \quad (2.21)$$

which express the  $Spin(1, 10)$  invariance of the Dirac matrices.

Equation in (2.9a) is just the  $(a) = (--) \equiv (0) - (\#)$  component of Eq. (2.21a) in the Dirac matrices realization in which  $\Gamma^0$  and  $\Gamma^{\#}$  are diagonal; the nine remaining  $\Gamma^I$  are off-diagonal. Eq. (2.9b) comes from the upper diagonal block of Eq. (2.21b). To complete the set of constraints defining the spinorial harmonics, we have to add the conditions expressing the invariance of the charge conjugation matrix  $C$ ,

$$VCV^T = C \quad , \quad V^T C^{-1} V = C^{-1} , \quad (2.22)$$

which give rise to the constraint (2.9c).

In a theory with the local  $SO(1, 1) \otimes SO(9)$  symmetry (2.12), (2.13), containing only one of the two sets of 16 constrained spinors (2.20), say  $v_{\alpha p}^{-}$ , these can be treated as homogeneous coordinates of the  $SO(1, 10)$  coset giving the celestial sphere  $S^9$ ; specifically (see [31])

$$\{v_{\alpha q}^{-}\} = \frac{Spin(1, 10)}{[Spin(1, 1) \otimes Spin(9)] \rtimes \mathbb{K}_9} = \mathbb{S}^9 , \quad (2.23)$$

where  $\mathbb{K}_9$  is the abelian subgroup of  $SO(1, 10)$  defined by<sup>7</sup>

$$\delta v_{\alpha q}^{-} = 0 , \quad \delta v_{\alpha q}^{+} = k^{++i} \gamma^i_{qp} v_{\alpha p}^{-} , \quad i = 1, \dots, 9 . \quad (2.24)$$

Our superparticle model contains just  $v_{\alpha q}^{-}$  and is invariant under  $SO(1, 1) \otimes Spin(9)$  transformations. Hence the harmonics sector of its configuration space parametrize  $S^9$  sphere.

#### 2.4.1 On harmonics and explicit parametrization of $SO(1, D-1)/H$ cosets

The vector harmonic variables, when constrained only by Eqs. (2.18), parametrize the eleven dimensional Lorentz group  $SO(1, 10)$ , Eq. (2.17). This, in principle, can be solved by expressing the harmonics in terms of 55 parameters  $l^{(a)(b)} = -l^{(b)(a)}$ ,  $U_m^{(a)} = U_m^{(a)}(l^{(b)(c)})$ ,

$$U_m^{(a)} = (u_m^{--}, u_m^{++}, u_m^i) = \delta_m^{(a)} + \eta_{m(b)} l^{(b)(a)} + \mathcal{O}(l^2) , \quad (2.25)$$

$$u_m^{\pm\pm} = \delta_m^{\pm\pm} - \eta_{m(b)} l^{\pm\pm(b)} + \mathcal{O}(l^2) , \quad u_m^i = \delta_m^i + \eta_{m(b)} l^{(b)i} + \mathcal{O}(l^2) ,$$

$$\delta_m^{\pm\pm} := \delta_m^0 \pm \delta_m^{\#} , \quad l^{(a)(b)} = -l^{(b)(a)} = \begin{pmatrix} 0 & -4l^{(0)} & l^{++j} \\ 4l^{(0)} & 0 & l^{--j} \\ -l^{++i} & -l^{--i} & l^{ij} \end{pmatrix} , \quad (2.26)$$

where we used the ‘light-like’ splitting  $(a) = ++, --, i$ ,  $i = 1, \dots, 9$ , so that

$$\eta_{(a)(b)} := \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\delta_{ij} \end{pmatrix} , \quad \eta^{(a)(b)} := \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -\delta_{ij} \end{pmatrix} , \quad (2.27)$$

---

<sup>7</sup>The  $\mathbb{K}_9$  symmetry (2.24) is tantamount to stating that the model contains only one,  $v_{\alpha p}^{-}$ , of the two sets of 16 constrained spinors  $(v_{\alpha q}^{-}, v_{\alpha q}^{+})$  in (2.20).

The same can be said about spinorial harmonics. Eqs. (2.21), (2.22) imply that spinorial harmonics parametrize the  $Spin(1, 10)$ -valued matrix providing the double covering of the  $SO(1, 10)$  group element (2.17) and, hence, that they can be expressed (up to the sign) through the same  $l^{(a)(b)} = -l^{(b)(a)}$  parameters,  $V_\alpha^{(\beta)} = \pm V_\alpha^{(\beta)}(l)$ ,

$$V_\alpha^{(\beta)}(l) = (v_{\alpha q}^-(l), v_{\alpha q}^+(l)) = \left( \delta_\alpha^{(\beta)} + \frac{1}{4} l^{(a)(b)} \Gamma_{(a)(b)\alpha}^{(\beta)} + \mathcal{O}(l^2) \right). \quad (2.28)$$

The identification of the harmonics with the coordinates of  $SO(1, 10)/H$  corresponds to setting to zero the  $H$  coordinates in the explicit expressions (2.25), (2.28). In our case with  $H = [SO(1, 1) \otimes SO(9)] \otimes \mathbb{K}_9$  this implies  $l^{(0)} = l^{ij} = l^{++j} = 0$  so that the  $SO(1, 10)$  and  $Spin(1, 10)$  matrices are constructed with the use of 9 parameters  $l^{--j}$ ,  $U_m^{(a)} = U_m^{(a)}(l^{--j})$ ,  $V_\alpha^{(\beta)} = V_\alpha^{(\beta)}(l^{--j})$ . These expressions are not so complicated and read

$$u_a^{--} = \delta_a^{--} + \delta_a^i l^{--i} + \frac{1}{2} \delta_a^{++} (l^{--j} l^{--j}), \quad u_a^{++} = \delta_a^{++}, \quad u_a^i = \delta_a^i + \frac{1}{2} \delta_a^{++} l^{--i}. \quad (2.29)$$

for the vector harmonics. The expressions for spinor harmonics are even simpler,

$$v_{\alpha q}^- = \delta_\alpha^{-q} + \frac{1}{2} l^{--i} \gamma_{qp}^i \delta_\alpha^{+q}, \quad v_{\alpha q}^+ = \delta_\alpha^{+q}. \quad (2.30)$$

The disadvantage of the above equations Eqs. (2.29), (2.30) with respect to the general Eqs. (2.25), (2.28), is that they are not Lorentz covariant; this follows from that they are the gauge fixed version of (2.25), (2.28) obtained with the use of  $[SO(1, 1) \otimes SO(9)] \otimes K_9$  symmetry.

Although the use of the explicit expressions (2.25), (2.28) is not practical (so that we even would not like to present them beyond the linear approximation; these explicit expressions can be found in [43]), it is convenient to keep in mind the mere fact of their existence. To make calculations we rather use the ‘admissible variation’ technique [7, 8] and/or, when the Hamiltonian mechanics is considered, the Dirac brackets for the constraints (2.17), (2.20) on the Harmonic variables and their conjugate. These Dirac brackets for  $U$ ,  $V$  and their momenta, which can be identified as Poisson brackets for  $l^{(a)(b)}$  in (2.17), (2.20) and its conjugate momentum, are discussed in the next section.

### 3 Hamiltonian mechanics of the D=11 superparticle in the spinor moving frame formulation and the BRST charge $\mathbb{Q}^{susy}$

In [3] we presented the supertwistor quantization of M0-branes. Albeit heuristic, it has the advantage of being simple, formulated in terms of physical variables (like the light-cone gauge quantization in [2]), and of being covariant (in contrast with [2]). Here we perform the complete Hamiltonian analysis of the dynamical system and consider its BRST quantization.

#### 3.1 Primary constraints of the D=11 superparticle model (M0-brane)

The primary constraints of the M0-brane in the spinor moving frame formulation (2.8) include the defining relations of the harmonic variables, Eqs. (2.9), plus other relations in (2.21), as well as

$$\Phi_a := P_a - \frac{1}{2} \rho^{++} u_a^{--} \approx 0 \quad \Leftrightarrow \quad \Phi_{\alpha\beta} := \Phi_a \Gamma_{\alpha\beta}^a = P_{\alpha\beta} - \rho^{++} v_{\alpha q}^- v_{\beta q}^- \approx 0, \quad (3.1)$$

$$d_\alpha := \pi_\alpha + i P_{\alpha\beta} \theta^\beta \approx 0, \quad \pi_\alpha := \frac{\partial L}{\partial \dot{\theta}^\alpha}, \quad P_m := \frac{\partial L}{\partial \dot{x}^m} \quad (3.2)$$

$$P_{++}^{(\rho)} := \frac{\partial L}{\partial \dot{\rho}^{++}} \approx 0, \quad (3.3)$$

and

$$P^{[u]}_{(a)}{}^m := \frac{\partial L}{\partial \dot{u}_m^{(a)}} \approx 0 \quad \text{or} \quad P^{[v]}_{(\alpha)}{}^\beta := \frac{\partial L}{\partial \dot{V}_\beta^{(\alpha)}} \approx 0. \quad (3.4)$$

The definition of the momenta

$$P_{\mathcal{N}} = \frac{\partial L}{\partial \dot{\mathcal{Z}}^{\mathcal{N}}} := \left( P_a, \pi_\alpha, P_{++}^{(\rho)}, P^{[u]}_{(a)}{}^m \text{ or } P^{[v]}_{(\alpha)}{}^\beta \right) \quad (3.5)$$

for the configuration space coordinates

$$\mathcal{Z}^{\mathcal{N}} := \left( x^a, \theta^\alpha, \rho^{++}, u_m^{(a)} \text{ or } V_\beta^{(\alpha)} \right) \quad (3.6)$$

determines the form of the (equal-proper-time) Poisson brackets  $[\dots, \dots]_{PB} := ([\dots, \dots]_{PB}, \{\dots, \dots\}_{PB})$

$$[\mathcal{Z}^{\mathcal{N}}, P_{\mathcal{N}'}]_{PB} := (-)^{\mathcal{N}} \delta_{\mathcal{N}'\mathcal{N}}, \quad [\dots, \dots]_{PB} := \frac{\partial \dots}{\partial \mathcal{Z}^{\mathcal{N}}} (-)^{\mathcal{N}} \frac{\partial \dots}{\partial P_{\mathcal{N}}} - \frac{\partial \dots}{\partial P_{\mathcal{N}}} \frac{\partial \dots}{\partial \mathcal{Z}^{\mathcal{N}}}. \quad (3.7)$$

The canonical Hamiltonian  $H_0$  is defined by

$$d\tau H_0 := d\mathcal{Z}^{\mathcal{N}} P_{\mathcal{N}} - d\tau L. \quad (3.8)$$

Since the canonical Hamiltonian of the massless superparticle is zero in the weak sense (*i.e.*, when constraints are taken into account [40]), its Hamiltonian analysis reduces to the analysis of the constraints. Following Dirac [40], we shall split the whole set of the constraints into first and second class ones and will deal with the second class constraints by using Dirac brackets.

To make the analysis more transparent it is convenient deal first with the second class constraints imposed on the harmonic variables.

### 3.2 Dirac brackets in Hamiltonian mechanics on the $SO(1, D-1)$ group manifold

Eqs. (2.25), (2.28) make manifest that the vector and the spinor Lorentz harmonics can be expressed through the same parameter  $l^{(a)(b)}$ . Hence one can, in principle, use the local  $l^{(a)(b)} = -l^{(b)(a)}$  coordinate in the configurational space ( $\mathcal{Z}^{\mathcal{N}} = (x^a, \theta^\alpha, \rho^{++}, l^{(a)(b)})$  in our case of massless superparticle) and develop the Hamiltonian mechanics using this variable and its conjugate momentum. This way is, however, technically involved.

Much more practical is to work with the whole set of Harmonic variables  $U$  and/or  $V$  and to take Eqs. (2.17), (2.20) into account by passing to the associated Dirac brackets. (This may be treated as an implicit use of Eqs. (2.25), (2.28) which, in terms of [40], would correspond to explicit solution of the corresponding second class constraints). It is more convenient to work in terms of vector harmonics; the corresponding Dirac brackets (as they actually coincide with the Poisson brackets for  $l$ ) can be then applied to the spinor harmonics as well.

When the harmonics enter as auxiliary variables, the primary constraints include the statement of vanishing of all the momentum conjugate to the vector harmonics,  $P_{(a)}{}^m = 0$  (Eq. (3.4)). This set of constraints can be easily split in a set of 55 constraints  $\mathbf{d}_{(a)(b)} := P_{(a)}{}^m U_{m(b)} - P_{(b)}{}^m U_{m(a)}$  and the 66 constraints  $\mathbf{K}_{(a)(b)} := P_{(a)}{}^m U_{m(b)} + P_{(b)}{}^m U_{m(a)}$ . These latter are manifestly second class ones as far as they are conjugate to the (also second class) 66 kinematical constraints (2.18),

$$\Xi^{(a)(b)} := U_m^{(a)} U^{m(b)} - \eta^{(a)(b)} \approx 0, \quad \mathbf{K}_{(a)(b)} := P_{(a)}{}^m U_{m(b)} + P_{(b)}{}^m U_{m(a)} \approx 0, \quad (3.9)$$

$$[\Xi^{(a)(b)}, \mathbf{K}_{(a')(b')}]_{PB} = 4\delta^{((a)}_{(a')} \delta^{(b)}_{(b')}]_{(b')} + 4\delta^{((a)}_{((a')} \Xi^{(b)}_{(b')}]_{(b')} \approx 4\delta^{((a)}_{(a')} \delta^{(b)}_{(b')}]_{(b')}, \quad (3.10)$$

while the 55 constraints  $\mathbf{d}_{(a)(b)} := P_{(a)}^m U_{m(b)} - P_{(b)}^m U_{m(a)}$  commute with the kinematical constraints  $\Xi^{(a)(b)}$ ,

$$\mathbf{d}_{(a)(b)} := P_{(a)}^m U_{m(b)} - P_{(b)}^m U_{m(a)} \approx 0, \quad [\Xi^{(a)(b)}, \mathbf{d}_{(a')(b')}]_{PB} = 0. \quad (3.11)$$

The brackets of these constraints represent the Lorentz group algebra while their brackets with  $\mathbf{K}_{(a)(b)}$  show that these are transformed as symmetric second rank tensor under the Lorentz group,<sup>8</sup>

$$[\mathbf{d}_{(a)(b)}, \mathbf{d}^{(c)(d)}]_{PB} = -4\delta_{[(a)}^{(c)} \mathbf{d}_{(b)]}^{(d)}, \quad [\mathbf{d}_{(a)(b)}, \mathbf{K}^{(c)(d)}]_{PB} = -4\delta_{[(a)}^{(c)} \mathbf{K}_{(b)]}^{(d)}. \quad (3.12)$$

Hence in the Lorentz harmonics sector of phase space one can define Dirac brackets

$$\begin{aligned} [\dots, \dots]_{DB_{harm}} &= [\dots, \dots]_{PB} - \frac{1}{4} [\dots, \mathbf{K}_{(a)(b)}]_{PB} [\Xi^{(a)(b)}, \dots]_{PB} \\ &\quad + \frac{1}{4} [\dots, \Xi^{(a)(b)}]_{PB} [\mathbf{K}_{(a)(b)}, \dots]_{PB} \end{aligned} \quad (3.13)$$

allowing us to use (2.18) and, moreover, all the 122 constraints (3.9) in the strong sense,

$$\Xi^{(a)(b)} := U_m^{(a)} U^{m(b)} - \eta^{(a)(b)} = 0, \quad \mathbf{K}_{(a)(b)} := P_{(a)}^m U_{m(b)} + P_{(b)}^m U_{m(a)} = 0. \quad (3.14)$$

Using (3.14) one sees that in the phase space sector that involves the harmonics  $U_{m(a)}$  and the ‘covariant momenta’  $\mathbf{d}_{(a)(b)} := P_{(a)}^m U_{m(b)} - P_{(b)}^m U_{m(a)}$ , but not the canonical momenta  $P_{(b)}^m$  themselves, the above defined Dirac brackets coincide with the Poisson brackets; in particular (see (3.12))

$$[\mathbf{d}_{(a)(b)}, \dots]_{DB_{harm}} = [\mathbf{d}_{(a)(b)}, \dots]_{PB}. \quad (3.15)$$

This reflects the fact that  $\mathbf{d}_{(a)(b)}$  provide a representation of the Lorentz group generators *i.e.* generate a parallel transport (‘translations’) along the Lorentz group manifold:  $[\mathbf{d}_{(a)(b)}, f(U)]_{PB} = (\frac{\partial}{\partial l^{(a)(b)}} + \dots) f(U(l))$  in terms of explicit parametrization in (2.25) (and (2.28) for spinorial harmonics,  $[\mathbf{d}_{(a)(b)}, f(V)]_{PB} = (\partial/\partial l^{(a)(b)} + \dots) f(V(l))$ ). The above described Dirac brackets give a convenient way to represent the Poisson brackets on the Lorentz  $SO(1, D-1)$  group manifold (which can also be formulated in terms of  $l^{(a)(b)} = -l^{(b)(a)}$  and its conjugate momentum).

This gives a reason for not distinguishing notationally these Dirac brackets  $[\dots, \dots]_{DB_{harm}}$  from the original Poisson brackets (3.7), denoting them also by  $[\dots, \dots]_{PB}$  or  $\{\dots, \dots\}_{PB}$  for the case of two fermionic constraints, and reserve the notation  $[\dots, \dots]_{DB}$ ,  $\{\dots, \dots\}_{PB}$  for the Dirac brackets allowing to resolve *all* the second class constraints for the M0-brane model.

### 3.3 Cartan forms and Hamiltonian mechanics on the Lorentz group manifold

The above Dirac brackets can be also applied [7] to calculations with the spinorial Lorentz harmonics. This is particularly important because the simple constraints on these variables, Eqs. (2.20), are reducible, and the irreducible constraints are not so easy to extract and to deal with. However, a relatively simple method to obtain the definite expressions for the above Dirac brackets and, more generally, to deal with the derivatives and variations of harmonic variables can be formulated using just the group-theoretical meaning of the harmonic variables (see [7] and also Appendix for more detail on this *admissible variation technique*).

Using the kinematic constraints (2.18) (first of Eqs. (3.9)) and (2.21), one can express the derivatives of both the vector and the spinor harmonics through the 55 Cartan forms,

$$\Omega^{(a)(b)} := U^{m(a)} dU_m^{(b)} = -\Omega^{(b)(a)} = \begin{pmatrix} 0 & -4\Omega^{(0)} & \Omega^{++j} \\ 4\Omega^{(0)} & 0 & \Omega^{--j} \\ -\Omega^{++i} & -\Omega^{--i} & \Omega^{ij} \end{pmatrix} \in so(1, 10). \quad (3.16)$$

<sup>8</sup>Furthermore, one can see that the Poisson brackets of two  $\mathbf{K}$ ’s close on  $\mathbf{d}_{(a)(b)}$ , so that the complete set of brackets of  $\mathbf{K}$  and  $\mathbf{d}_{(a)(b)}$  constraints represent  $gl(D, \mathbb{R})$ ; the  $\mathbf{K}_{(a)(b)}$  constraints correspond to the  $\frac{GL(D, \mathbb{R})}{SO(1, D-1)}$  coset generators.

Indeed, the equation

$$dU_m^{(a)} = U_{m(b)} \Omega^{(b)(a)} \quad (3.17)$$

is just equivalent to the definition of the Cartan forms, Eq. (3.16), when (2.18) (or equivalent (2.19)) is taken into account. As, according to (2.21), the spinorial harmonic matrix  $V$  provides the spinorial representation of the  $Spin(1, D-1)$  element  $g$  which correspond to the Lorentz rotation  $U$ , its derivative can be expressed through the same Cartan form  $g^{-1}dg = \frac{1}{2}\Omega^{(a)(b)}\mathbb{T}_{(a)(b)}$ , but with  $\mathbb{T}_{(a)(b)} = \frac{1}{2}\Gamma_{(a)(b)}$  instead of  $\mathbb{T}_{(a)(b)(c)}^{(d)} = 2\eta_{(c)[(a)}\delta_{(b)]}^{(d)}$  giving rise to Eq. (3.16),

$$V^{-1}dV = \frac{1}{4}\Omega^{(a)(b)}\Gamma_{(a)(b)} \in spin(1, 10), \quad \Omega^{(a)(b)} := U^{m(a)}dU_m^{(b)} \quad (3.18)$$

Eq. (3.18) can be equivalently written in the form of  $dV = \frac{1}{4}\Omega^{(a)(b)}V\Gamma_{(a)(b)}$ . This equation implies, in particular, the following expression for the differential  $dv_{\alpha\bar{q}}^-$  of the harmonics  $v_{\alpha\bar{q}}^-$  entering the action (2.8):

$$dv_q^- = -\Omega^{(0)}v_q^- - \frac{1}{4}\Omega^{ij}v_p^-\gamma_{pq}^{ij} + \frac{1}{2}\Omega^{-i}\gamma_{qp}^i v_p^+ \quad (3.19)$$

The particular  $((a) = --)$  case of Eq. (3.17) gives

$$du_m^{--} = -2u_m^{--}\Omega^{(0)} + u_m^i\Omega^{-i} \quad (3.20)$$

for the derivative of the only vector harmonics that appear explicitly in the action (2.8). Notice that (3.19) and (3.20) do not contain the Cartan form  $\Omega^{++i}$ , corresponding to the abelian  $\mathbb{K}_9$  subgroup (see Eq. (2.24)) of  $SO(1, 10)$  parametrized by the harmonics. This actually reflects the  $\mathbb{K}_9$  gauge invariance of the action (2.8), allowing, together with its manifest  $SO(1, 1)$  and  $SO(9)$  invariance, to identify the relevant harmonics  $u_m^{--}$  and  $v_{\alpha\bar{q}}^-$  with the homogeneous coordinates of  $\mathbb{S}^9$ , Eq. (2.23).

When the Hamiltonian formalism for a dynamical system involving harmonic variables is considered, one can use, as above, the standard way to define hamiltonian,  $H_0 = \partial_\tau u P^{[u]} + \dots - L$  or  $H_0 = \partial_\tau V P^{[v]} + \dots - L$ , Eq. (3.8), and introduce the Dirac brackets (3.13). Alternatively one can use Eqs. (3.17), (3.18) in the above expressions for  $H_0$ , or better for  $d\tau H_0$ , and, in such a way, to arrive at the Hamiltonian of the form

$$d\tau H_0 = -\frac{1}{2}\Omega^{(a)(b)}\mathbf{d}_{(a)(b)} + \dots - d\tau L, \quad \mathbf{d}_{(a)(b)} := u_{(b)}^m P_{(a)m} - u_{(a)}^m P_{(b)m} \quad (3.21)$$

containing the Cartan form (3.16) and the ‘covariant momentum’  $\mathbf{d}_{(a)(b)}$  (see (3.11)) instead of  $dU$  or  $dV$  and its conjugate momentum.

Such a Hamiltonian can be thought of as the one with the kinematical constraints solved in terms of the independent parameter  $l$  ( $U = U(l)$ ,  $V = V(l)$ , see Eqs. (2.25), (2.28)), but, as we see, one does not need using the explicit form of such a solution. In particular, to find the Poisson bracket of the ‘covariant momentum’  $\mathbf{d}_{(a)(b)}$  with harmonics one can just use the general form of the Hamiltonian equations  $\dot{U} := [U, H_0]_{PB}$  or  $\dot{V} := [V, H_0]_{PB}$ , and the explicit expression for the Cartan form, (3.16) and (3.18) for the case of spinor Harmonics. Indeed, for the vector harmonic  $dU_m^{(a)} := d\tau [U_m^{(a)}, H_0]_{PB} = -\frac{1}{2}\Omega^{(c)(d)}[U_m^{(a)}, \mathbf{d}_{(c)(d)}]_{PB} = -\frac{1}{2}dU_n^{(d)}[U_m^{(a)}, \mathbf{d}_{(c)(d)}]_{PB} U^{n(c)}$  implies

$$[\mathbf{d}_{(a)(b)}, U_m^{(a')}]_{PB} = 2U_{m[(a)}\delta_{(b)]}^{(a')} \quad (3.22)$$

Making the similar calculation with the spinor harmonics, one finds

$$[\mathbf{d}_{(a)(b)}, V_\alpha^{(\beta)}]_{PB} = \frac{1}{2}V_\alpha^{(\gamma)}\Gamma_{(a)(b)(\gamma)}^{(\beta)}\delta_{(d)}^{(a')} \quad (3.23)$$

Then, calculating the Poisson bracket of (3.22) and  $\mathbf{d}_{(a)(b)}$ , and using the Jacobi identities for the Poisson brackets we find the first of Eqs. (3.12)

$$[\mathbf{d}_{(a)(b)} , \mathbf{d}^{(c)(d)}]_{PB} = -4\delta_{[(a)}^{[c)} \mathbf{d}_{(b)]}^{(d)]} , \quad (3.24)$$

which implies that  $\mathbf{d}_{(a)(b)}$  are the Lorentz group generators.

Thus, using the kinematical constraints (2.18) and/or (2.21) in the strong sense we also can easily construct the canonical Hamiltonian and the Poisson brackets directly on the  $SO(1, D-1)$  group manifold, thus overcoming the stage of introducing the Dirac brackets (3.13) and escaping the use of explicit parametrization (2.25), (2.28).

### 3.4 Canonical Hamiltonian and Poisson/Dirac brackets of the M0-brane model

The discussion and equations of the previous section hold for Hamiltonian mechanics on any space including Lorentz group  $SO(1, D-1)$  or its coset  $SO(1, D-1)/H$  as a subspace. The harmonics used in the twistor-like formulations of super- $p$ -branes with  $p \geq 1$  [6, 7, 8] are homogeneous coordinates of the coset with  $H = SO(1, p) \otimes SO(D-p-1)$ . The case of massless superparticle ( $p=0$ ) is special. Here the  $H = [SO(1, 1) \otimes SO(D-2)] \ltimes \mathbb{K}_{D-2}$  is the Borel (maximal compact) subgroup of  $SO(1, D-1)$ . In this case (as well as in the string case [6, 7]) one uses the  $H$ -covariant splitting (3.16) to arrive at

$$\begin{aligned} d\tau H_0 := & -\frac{1}{2}\Omega^{--i}\mathbf{d}^{++i} - \frac{1}{2}\Omega^{++i}\mathbf{d}^{--i} - \Omega^{(0)}\mathbf{d}^{(0)} + \frac{1}{2}\Omega^{ij}\mathbf{d}^{ij} + \\ & + dx^a P_a + d\theta^\alpha \pi_\alpha + d\rho^{++} P_{++}^{(\rho)} - d\tau L . \end{aligned} \quad (3.25)$$

Then the Poisson/Dirac brackets can be defined by the following set of non-zero relations (see (3.7))

$$[P_a , x^b]_{PB} = -\delta_a^b , \quad \{\pi_\alpha , \theta^\beta\}_{PB} = -\delta_\alpha^\beta , \quad [P_{++}^{(\rho)} , \rho^{++}]_{PB} = -1 , \quad (3.26)$$

as well as Eqs. (3.22), (3.23) and the Lorentz group algebra (3.24) which splits as

$$\begin{aligned} [\mathbf{d}^{++i} , \mathbf{d}^{--j}]_{PB} &= 2\mathbf{d}^{ij} + \mathbf{d}^{(0)}\delta^{ij} , & [\mathbf{d}^{(0)} , \mathbf{d}^{\pm\pm i}]_{PB} &= \pm 2\mathbf{d}^{\pm\pm i} , \\ [\mathbf{d}^{ij} , \mathbf{d}^{\pm\pm k}]_{PB} &= 2\mathbf{d}^{\pm\pm[i}\delta^{j]k} , & [\mathbf{d}^{ij} , \mathbf{d}^{kl}]_{PB} &= 2\mathbf{d}^{k[i}\delta^{j]l} - 2\mathbf{d}^{l[i}\delta^{j]k} . \end{aligned} \quad (3.27)$$

The splitting  $\mathbf{d}_{(a)(b)} = (\mathbf{d}^{(0)} , \mathbf{d}^{\pm\pm j} , \mathbf{d}^{ij})$  of the  $SO(1, 10)$  generators (see 3.16) is invariant under  $SO(1, 1) \otimes SO(9)$  symmetry the generators of which are represented by  $\mathbf{d}^{(0)} , \mathbf{d}^{ij}$ . The set of remaining generators  $\mathbf{d}^{++j} , \mathbf{d}^{--j}$  can be conventionally split on two Abelian subsets, one, say  $\mathbf{d}^{--j}$ , representing the  $\mathbb{K}_9$  generator, and other,  $\mathbf{d}^{++j}$ , corresponding to the  $SO(1, 10)/SO(1, 1) \otimes SO(9)] \ltimes \mathbb{K}_9$  coset.

The split form of Eqs. (3.22), (3.23) include

$$\begin{aligned} [\mathbf{d}^{(0)} , u_m^{--}]_{PB} &= -2u_m^{--} , & [\mathbf{d}^{--i} , u_m^{--}]_{PB} &= 0 , & [\mathbf{d}^{++i} , u_m^{--}]_{PB} &= 2u_m^i , \\ & & [\mathbf{d}^{ij} , u_m^{--}]_{PB} &= 0 , \end{aligned} \quad (3.28)$$

$$\begin{aligned} [\mathbf{d}^{(0)} , v_q^-]_{PB} &= -v_q^- , & [\mathbf{d}^{--i} , v_q^-]_{PB} &= 0 , & [\mathbf{d}^{++i} , v_q^-]_{PB} &= \gamma_{qp}^i v_p^+ , \\ & & [\mathbf{d}^{ij} , v_q^-]_{PB} &= \frac{1}{2}v_p^- \gamma_{pq}^{ij} . \end{aligned} \quad (3.29)$$

$$\begin{aligned} [\mathbf{d}^{(0)} , v_q^+]_{PB} &= v_q^+ , & [\mathbf{d}^{--i} , v_q^+]_{PB} &= \gamma_{qp}^i v_p^- , & [\mathbf{d}^{++i} , v_q^+]_{PB} &= 0 , \\ & & [\mathbf{d}^{ij} , v_q^+]_{PB} &= \frac{1}{2}v_p^+ \gamma_{pq}^{ij} , \end{aligned} \quad (3.30)$$

and the relations for the brackets of  $\mathbf{d}_{(a)(b)}$  with  $u_m^{++}$  and  $u_m^i$  vectors, which are not needed in this paper. All these relations can be collected in

$$[\mathbf{d}^{(a)(b)} , U]_{PB} := \mathbb{D}^{(a)(b)} U , \quad [\mathbf{d}^{(a)(b)} , V]_{PB} := \mathbb{D}^{(a)(b)} V , \quad (3.31)$$

where  $\mathbb{D}_{(a)(b)} = (\mathbb{D}^{\pm\pm i}, \mathbb{D}^{ij}, \mathbb{D}^{(0)})$  are the covariant harmonic derivatives which provide the differential operator representation for the Lorentz group generators ( $\mathbb{D}_{(a)(b)} = \partial/\partial l^{(a)(b)} + \dots$  in terms of explicit parametrization) which are defined by the decomposition of the differential on the Cartan forms (3.16)<sup>9</sup>,

$$d := \frac{1}{2}\Omega^{(a)(b)}\mathbb{D}_{(a)(b)} =: \Omega^{(0)}\mathbb{D}^{(0)} + \frac{1}{2}\Omega^{++i}\mathbb{D}^{--i} + \frac{1}{2}\Omega^{--i}\mathbb{D}^{++i} - \frac{1}{2}\Omega^{ij}\mathbb{D}^{ij}. \quad (3.32)$$

### 3.5 Second class constraints of the D=11 superparticle model

With the Poisson/Dirac brackets (3.26)–(3.30), the phase space  $(Z^{\mathcal{N}}, P_{\mathcal{N}})$  of our superparticle model includes, for the moment, the  $Spin(1, 10)$  group manifold, parametrized by harmonics, and the corresponding momentum space parametrized by the non-commutative generalized momenta  $\mathbf{d}_{(a)(b)}$  of Eqs. (3.21), (3.24), (3.31). In all we have<sup>10</sup>

$$P_{\mathcal{N}} = \left( P_a, \pi_\alpha, P_{++}^{(\rho)}, \mathbf{d}_{(a)(b)} \right), \quad \mathcal{Z}^{\mathcal{N}} := \left( x^a, \theta^\alpha, \rho^{++}, V_\beta^{(\alpha)} \right) \in Spin(1, 10) \quad (3.33)$$

This phase space (3.33) is restricted by the constraints (3.1), (3.2), (3.3) and

$$\mathbf{d}_{(a)(b)} \approx 0 \quad \Leftrightarrow \quad \begin{cases} \mathbf{d}^{(0)} \approx 0, & \mathbf{d}^{ij} \approx 0, & \mathbf{d}^{--i} \approx 0, \\ \mathbf{d}^{++i} \approx 0 \end{cases} \quad (3.34)$$

for the non-commutative momentum of the  $Spin(1, 10)$  group valued spinor moving frame variables  $V \in Spin(1, 10)$  [instead of the ‘original’ (3.4) for an apparently unrestricted  $V$  matrices]. The algebra of primary constraints (3.1), (3.2), (3.3) and (3.34) is characterized by the following nonvanishing brackets

$$[\Phi_a, P_{++}^{[\rho]}]_{PB} = -\frac{1}{2}u_a^{--}, \quad [\Phi_a, \mathbf{d}^{(0)}]_{PB} = -\rho^{++}u_a^{--}, \quad [\Phi_a, \mathbf{d}^{++i}]_{PB} = -\rho^{++}u_a^i, \quad (3.35)$$

$$\{d_\alpha, d_\beta\}_{PB} = -2iP_{\alpha\beta} \equiv -2i\Phi_{\alpha\beta} - 2i\rho^{++}v_{\alpha q}^-v_{\beta q}^-, \quad (3.36)$$

and the Lorentz algebra relations (3.27). This allows us to find the following *fermionic and bosonic second class constraints*, the latter split in mutually conjugate pairs

$$\begin{aligned} d_q^+ &:= v_q^{+\alpha}d_\alpha \approx 0, & \{d_q^+, d_p^+\}_{PB} &= -2i\rho^{++}\delta_{pq}, \\ u^{a++}\Phi_a &\approx 0, \quad P_{++}^{[\rho]} \approx 0, & [u^{a++}\Phi_a, P_{++}^{[\rho]}]_{PB} &= -1, \\ u^{ai}\Phi_a &\approx 0, \quad \mathbf{d}^{++j} \approx 0, & [u^{ai}\Phi_a, \mathbf{d}^{++j}]_{PB} &= -\rho^{++}. \end{aligned} \quad (3.37)$$

Here  $v_q^{+\alpha}$  is an element of the inverse spinor moving frame matrix  $V^{-1}{}_{(\beta)}^\alpha = (v_q^{+\alpha}, v_q^{-\alpha}) \in Spin(1, 10)$  which obeys  $v_q^{+\alpha}v_{\alpha q}^+ = 0$  and  $v_q^{+\alpha}v_{\alpha q}^- = \delta_{qp}$ . In  $D=11$  (as in the other cases when the charge conjugation matrix exists) this is expressed through the original spinor harmonics with the help of Eqs. (2.22),

$$D = 11 : \quad v_q^{\pm\alpha} = \pm i C^{\alpha\beta} v_{\beta q}^\pm \quad (3.38)$$

(notice that the  $D = 11$  charge conjugation matrix is imaginary in our ‘mostly minus’ signature).

<sup>9</sup> The minus signs in (3.25) are chosen to provide the plus sign in (3.31).

<sup>10</sup> Here it is convenient to consider vector harmonics  $U_m^{(a)} \in SO(1, 10)$  as composites of the spinorial ones,  $V_\beta^{(\alpha)} \in Spin(1, 10)$ , defined by the gamma-trace parts of Eqs. (2.21),  $U_m^{(a)} = \frac{1}{32} tr V \Gamma^{(a)} V^T \tilde{\Gamma}_m$ .



Introducing the Dirac brackets

$$\begin{aligned}
[\dots, \dots]_{DB} = & [\dots, \dots]_{PB} + [\dots, P_{++}^{[\rho]}]_{PB} \cdot [(u^{++}P - \rho^{++}), \dots]_{PB} - \\
& - [\dots, (u^{++}P - \rho^{++})]_{PB} \cdot [P_{++}^{[\rho]}, \dots]_{PB} - \\
& - [\dots, u^j P]_{PB} \frac{1}{\rho^{++}} [\mathbf{d}^{++j}, \dots]_{PB} + [\dots, \mathbf{d}^{++j}]_{PB} \frac{1}{\rho^{++}} [u^j P, \dots]_{PB} - \\
& - [\dots, d_q^+]_{PB} \frac{i}{2\rho^{++}} [d_q^+, \dots]_{PB}, \tag{3.39}
\end{aligned}$$

one can treat the second class constraints as the strong relations

$$d_q^+ := v_q^{+\alpha} d_\alpha = 0; \quad \rho^{++} = u^{a++} P_a, \quad P_{++}^{[\rho]} = 0; \quad u^{ai} P_a = 0, \quad \mathbf{d}^{++j} = 0. \tag{3.40}$$

### 3.6 First class constraints and their (nonlinear) algebra

The remaining constraints are

$$d_q^- := v_q^{-\alpha} d_\alpha \approx 0, \quad u^{a--} \Phi_a = u^{a--} P_a =: P^{--} \approx 0, \tag{3.41}$$

$$\mathbf{d}^{ij} \approx 0, \quad \mathbf{d}^{(0)} \approx 0, \quad \mathbf{d}^{-i} \approx 0. \tag{3.42}$$

They give rise to the first class constraints. Namely, the Dirac bracket algebra of the constraints (3.41), (3.42) is closed and contains the following nonvanishing brackets

$$[\mathbf{d}^{ij}, \mathbf{d}^{kl}]_{DB} = 4\mathbf{d}^{[k|[i}\delta^{j]l]}, \quad [\mathbf{d}^{ij}, \mathbf{d}^{--k}]_{DB} = 2\mathbf{d}^{--[i}\delta^{j]k}, \quad [\mathbf{d}^{(0)}, \mathbf{d}^{\pm\pm i}]_{DB} = \pm 2\mathbf{d}^{\pm\pm i}, \tag{3.43}$$

$$\boxed{[\mathbf{d}^{--i}, \mathbf{d}^{--j}]_{DB} = \frac{i}{2\rho^{++}} d_q^{-ij} d_p^{-}}, \tag{3.44}$$

$$[\mathbf{d}^{ij}, d_p^-]_{DB} = -\frac{1}{2}\gamma_{pq}^{ij} d_q^-, \quad [\mathbf{d}^{(0)}, d_p^-]_{DB} = -d_q^-, \quad [\mathbf{d}^{(0)}, P^{--}]_{DB} = -2P^{--}, \tag{3.45}$$

$$\boxed{\{d_q^-, d_p^-\}_{DB} = -2i\delta_{qp} P^{--}}. \tag{3.46}$$

Notice that the right hand side of Eq. (3.44) includes the product of the two fermionic first class constraint and, hence, implies moving outside the Lie algebra (to the enveloping algebra)<sup>11</sup>. If this term were absent, one would state that the first class constraints (3.42) generated  $H = SO(1, 1) \otimes SO(9) \ltimes K_9$  group symmetry, and the whole gauge symmetry would be described by its semidirect product (see (3.45))  $H \ltimes \Sigma^{(1|16)}$  with the  $d = 1, N = 16$  supersymmetry group  $\Sigma^{(1|16)}$  of the  $\kappa$ -symmetry and  $b$ -symmetry, Eqs. (3.41), (3.46). Then the actual algebra of Eqs. (3.43), (3.44), (3.45), (3.46) is a ‘*generalized W-deformation*’ of the Lie superalgebra of this semidirect product  $[SO(1, 1) \otimes SO(9)] \ltimes K_9 \ltimes \Sigma^{(1|16)}$ . The role of the constant parameter for the standard deformation here is taken by the function  $\frac{1}{\rho^{++}}$  (hence the name *generalized* for this ‘W-deformation’). However, although momentum  $P^{++} = u^{a++} P_a$  is a dynamical variable, it has vanishing Dirac brackets with all the first class constraints.

One may guess that the complete BRST charge  $\mathbb{Q}$  for the algebra of the first class constraints (3.46) is quite complicated and its use is not too practical. Following the pragmatic spirit of

<sup>11</sup> One may also think of this as an analogy of the very well known phenomenon of the non-commutativity of the bosonic spacetime coordinates of the superparticle which appears in standard formulation [45] after transition to the Dirac brackets for the second class constraints; see also the second reference in [36]. There the Dirac brackets of two bosonic coordinates are proportional to the product of two Grassmann coordinates [45, 36]. In four dimensions such a noncommutativity is overcome by passing to the so called chiral basis of  $D = 4$  superspace the imaginary part of the bosonic coordinate of which is given by the Grassmann coordinates bilinear. The use of the Gupta-Bleuler technique [45, 46] also helps. The appearance of a nonlinear algebra of constraints was also observed for the twistor-like formulation of  $D=4$  null superstring and null-supermembranes in [47]. Notice finally that among the ‘nonlinear algebras’, the most popular are the W-algebra intensively studied some years ago (see *e.g.* [49] and reference therein).

the pure spinor approach [10, 26], it is tempting to take care of the constraints corresponding to the (deformed)  $[SO(1,1) \otimes SO(9)] \ltimes K_9$  part of the gauge symmetries in a different manner, by imposing them as conditions on the wavefunctions in quantum theory, and to leave with a short and fine BRST charge corresponding to the supersymmetry algebra (3.46) of the  $\kappa$ -symmetry and the  $b$ -symmetry generators.

However, the appearance of the deformation given by the product of the fermionic first class constraints in the *r.h.s.* of Eq. (3.44) might produce doubts on the consistency of such a prescription. Indeed, imposing, for instance, the deformed (now non-Abelian)  $K_9$  constraint  $\mathbf{d}^{--i}$  as a condition on the wave function in quantum theory,  $\widehat{\mathbf{d}}^{--i} \Phi = 0$ , one should also impose by consistency the condition,  $[\widehat{\mathbf{d}}^{--i}, \widehat{\mathbf{d}}^{--j}] \Phi = 0$  which implies  $\gamma_{qp}^{ij} \widehat{d}_q^- \widehat{d}_p^- \Phi = 0$ .

To clarify the situation with the BRST quantization of the nonlinear algebra (3.43)–(3.46) and its possible simplification we begin with studying the BRST charge  $\mathbb{Q}'$  corresponding to the subalgebra of  $\kappa$ -,  $b$ - and  $K_9$ -symmetry generators,  $d_q^-$ ,  $P^{--}$  and  $\mathbf{d}^{--i}$ .

### 3.7 BRST charge for a nonlinear sub(super)algebra of the first class constraints

The sub-superalgebra of the  $\kappa$ -,  $b$ - and the deformed  $K_9$ -symmetry generators,  $d_q^-$ ,  $P^{--}$  and  $\mathbf{d}^{--i}$  is described by Eqs. (3.44) and (3.46) plus vanishing brackets for the rest,

$$[\mathbf{d}^{--i}, \mathbf{d}^{--j}]_{DB} = \frac{i}{2P^{++}} d_q^- \gamma_{qp}^{ij} d_p^- \quad (a), \quad \{d_q^-, d_p^-\}_{DB} = -2i\delta_{qp} P^{--} \quad (b). \quad (3.47)$$

It is obtained from (3.43)–(3.46) by setting the generators of  $SO(9) \otimes SO(1,1)$  equal to zero,  $\mathbf{d}^{ij} = 0$  and  $\mathbf{d}^{(0)} = 0$ . Notice that, when acting on the space of  $SO(9) \otimes SO(1,1)$  invariant functions, the full BRST charge  $\mathbb{Q}$  of our  $D = 11$  superparticle reduces to the BRST charge of the algebra (3.47). In the quantum theory such an algebra reduction can be realized by imposing  $\mathbf{d}^{ij}$  and  $\mathbf{d}^{(0)}$  as conditions on the state vectors  $\widehat{\mathbf{d}}^{ij} \Phi = 0$  and  $\widehat{\mathbf{d}}^{(0)} \Phi = 0$ . This specifies the wavefunction dependence on the harmonics making it a function on the non-compact coset  $SO(1,10)/[SO(9) \times SO(1,1)]$  (dependence on  $l^{\pm i}$  parameters only in the case of explicit parametrization (2.25), (2.28)).

We denote the BRST charge corresponding to the non-linear superalgebra (3.47) by  $\mathbb{Q}'$  which reflects the fact that it gives only a part of the full BRST charge describing the complete gauge symmetry algebra (3.43)–(3.46) of the M0-brane in spinor moving frame formulation. The master equation

$$\{\mathbb{Q}', \mathbb{Q}'\}_{DB} = 0 \quad (3.48)$$

has the solution

$$\begin{aligned} \mathbb{Q}' = & \lambda_q^+ d_q^- + c^{++} P^{--} + c^{++j} \mathbf{d}^{--j} - i\lambda_q^+ \lambda_q^+ \pi_{++}^{[c]} + \frac{i}{2P^{++}} c^{++i} c^{++j} d_q^- \gamma_{qp}^{ij} P_p^{-[\lambda]} + \\ & + \frac{1}{P^{++}} c^{++i} c^{++j} \lambda_q^+ \gamma_{qp}^{ij} P_p^{-[\lambda]} \pi_{++}^{[c]} - \frac{i}{4(P^{++})^2} c^{++i} c^{++j} c^{++k} c^{++l} P_q^{-[\lambda]} \gamma_{qp}^{ijkl} P_p^{-[\lambda]} \pi_{++}^{[c]}. \end{aligned} \quad (3.49)$$

Here  $\lambda_q^+$  is the bosonic ghost for the fermionic  $\kappa$ -symmetry gauge transformations,  $c^{++}$  and  $c^{++j}$  are the fermionic ghosts for the bosonic  $b$ -symmetry and deformed  $\mathbb{K}_9$  symmetry transformations, and  $P_q^{-[\lambda]}$ ,  $\pi_{++}^{[c]}$  are the (bosonic and fermionic) ghost momenta conjugate to  $\lambda_q^+$  and  $c^{++}$ ,

$$[\lambda_q^+, P_p^{-[\lambda]}]_{DB} = \delta_{qp}, \quad \{c^{++}, \pi_{++}^{[c]}\}_{DB} = -1, \quad \{c^{++i}, \pi_{++j}^{[c]}\}_{DB} = -\delta_j^i. \quad (3.50)$$

Notice that the fermionic ghost momentum  $\pi_{++j}^{[c]}$  conjugate to  $c^{++j}$  does not enter  $\mathbb{Q}'$  (3.49).

The  $\mathbb{Q}'$  of Eq. (3.49) is the third rank BRST charge in the sense that the series stops on the third degree in the ghost momenta  $P_p^{-[\lambda]}$ ,  $\pi_{++}^{[c]}$ . Technically, the decomposition stops due to nilpotency of  $\pi_{++}^{[c]}$ . The nilpotency of the BRST charge (3.49) is preserved in the quantum theory,  $(\mathbb{Q}')^2 = 0$ , as far as no products of noncommuting operators (like *e.g.*  $\lambda_q^+ P_q^{-[\lambda]}$ ) appear in the calculation of  $(\mathbb{Q}')^2$ .

### 3.8 The further reduced BRST charge $\mathbb{Q}^{susy}$

The (already restricted) BRST charge (3.49) is (still) too much complicated to discuss it as a counterpart of (or as an alternative to) the Berkovits pure spinor BRST charge. A (further) reduction looks necessary. To this end let us notice that  $\mathbb{Q}'$  of Eq. (3.49) can be presented as a sum

$$\mathbb{Q}' = \mathbb{Q}^{susy} + c^{++j} \tilde{\mathbf{d}}^{--j}, \quad (3.51)$$

of the much simpler operator

$$\boxed{\mathbb{Q}^{susy} = \lambda_q^+ d_q^- + c^{++} P^{--} - i \lambda_q^+ \lambda_q^+ \pi_{++}^{[c]}}, \quad \{\mathbb{Q}^{susy}, \mathbb{Q}^{susy}\}_{DB} = 0, \quad (3.52)$$

and the term containing the  $c^{++j}$  ghost fields. The operator (3.52) can be identified as BRST charge corresponding to the  $d = 1$ ,  $N = 16$  supersymmetry algebra

$$\{d_q^-, d_p^-\}_{DB} = -2iP^{--}, \quad [P^{--}, d_p^-]_{DB} = 0, \quad [P^{--}, P^{--}]_{DB} \equiv 0. \quad (3.53)$$

of the  $\kappa$ - and  $b$ -symmetry generators (3.53). The second term in (3.51),  $c^{++j} \tilde{\mathbf{d}}^{--j}$ , contains the deformed  $K_9$  generator modified by additional ghost contributions,

$$\begin{aligned} \tilde{\mathbf{d}}^{--i} &= \mathbf{d}^{--i} + \frac{i}{2P^{++}} c^{++j} d_q^- \gamma_{qp}^{ij} P_p^{-[\lambda]} + \frac{1}{P^{++}} c^{++j} \lambda_q^+ \gamma_{qp}^{ij} P_p^{-[\lambda]} \pi_{++}^{[c]} - \\ &\quad - \frac{i}{4(P^{++})^2} c^{++j} c^{++k} c^{++l} P_q^{-[\lambda]} \gamma_{qp}^{ijkl} P_p^{-[\lambda]} \pi_{++}^{[c]}. \end{aligned} \quad (3.54)$$

The ‘nilpotency’ of the  $\mathbb{Q}^{susy}$  (3.52) ( $\{\mathbb{Q}^{susy}, \mathbb{Q}^{susy}\}_{DB} = 0$ ) guaranties the consistency of the reduction of the  $\mathbb{Q}'$ -cohomology problem to the  $\mathbb{Q}^{susy}$ -cohomology. For the classical BRST charge such a reduction can be reached just by setting the  $K_9$  ghost equal to zero,  $c^{++j} = 0$ . In classical mechanics one can consider this reduction as a result of the gauge fixing, *e.g.* in the explicit parametrization (2.25), (2.28) by setting  $l^{++i} = 0$  and (as  $l^{ij} = l^{(0)} = 0$  can be fixed by  $SO(1,1) \otimes SO(9)$  transformations) expressing all the harmonics in terms of nine parameters  $l^{--i}$  (related to the projective coordinates of the  $S^9$  sphere) as in Eqs. (2.29), (2.30).

Although technical, the question of how to realize a counterpart of such a classical gauge fixing in quantum description looks quite interesting. The problem is whether in this way one arrives just at scalar functions on  $S^9 = SO(1,10)/[SO(1,1) \otimes SO(9)] \ltimes K_9$ , or the interplay of the  $v_q^+$  (or  $u_m^{++}$ ,  $u_m^i$ ) harmonics and the  $K_9$  ghost  $c^{++j}$  may result in wavefunctions transforming nontrivially under  $SO(1,1) \otimes SO(9)$  (a counterpart of the effect of the  $D=4$  helicity appearance in the quantization of  $D=4$  superparticle, see [5] and refs. herein). Such an interplay could appear, *e.g.* when one imposes the quantum counterpart of the deformed  $K_9$  constraints modified by ghost contribution (3.54) on the wavefunctions. However, this interesting problem is out of the score of the present paper devoted to a search for the origin and geometric meaning of the Berkovits approach in the frame of spinor moving frame formulation of (presently) M0-brane.

Thus, let us accept, following the pragmatic spirit of the pure spinor approach [10], the simple prescription of the reduction of the first class constraint Dirac brackets algebra down to the  $d = 1$   $N = 16$  supersymmetry algebra of  $\kappa$ -symmetry and  $b$ -symmetry, Eq. (3.46) (taking care of other constraints in a different manner), which implies the reduction of  $\mathbb{Q}'$  to the much simpler  $\mathbb{Q}^{susy}$ , and let us turn to the study of the cohomology problem for the BRST charge  $\mathbb{Q}^{susy}$  (3.52).

## 4 BRST quantization of the D=11 superparticle. Cohomology of $\mathbb{Q}^{susy}$ and the origin of the complexity of the Berkovits approach

### 4.1 Quantum BRST charge $\mathbb{Q}^{susy}$

It is practical, omitting the overall  $\pm i$  factor, to write the quantum BRST charge obtained from (3.52) as

$$\mathbb{Q}^{susy} = \lambda_q^+ D_q^- + ic^{++} \partial_{++} - \lambda_q^+ \lambda_q^+ \frac{\partial}{\partial c^{++}}, \quad \{\mathbb{Q}^{susy}, \mathbb{Q}^{susy}\} = 0, \quad (4.1)$$

where the quantum operators  $D_q^-$  and  $\partial_{++}$ , associated with  $d_q^-$  and  $P_{++}$ , obey the  $d = 1, n = 16$  supersymmetry algebra (*cf.* (3.46))

$$\{D_p^-, D_q^-\} = 2i\delta_{qp} \partial_{++}, \quad [\partial_{++}, D_p^-] = 0. \quad (4.2)$$

The quantum BRST operator  $\mathbb{Q}^{susy}$  (4.1) should act on the space of wavefunctions that depend on the physical (gauge invariant) variables and on a number of variables which transform nontrivially under the action of generators  $\partial_{++}$ ,  $D_q^-$  (in general case, the variables of a model cannot be split covariantly on gauge invariant and pure gauge ones, but for our model this is actually possible, see Sec. 5). It is convenient to use a realization of  $\partial_{++}$ ,  $D_q^-$  as differential operators on the  $1 + 16$  dimensional superspace  $W^{(1|16)}$  of coordinates  $(x^{++}, \theta_q^+)$ ,

$$D_q^- = \partial_{+q} + i\theta_q^+ \partial_{++}, \quad \partial_{++} := \frac{\partial}{\partial x^{++}}, \quad \partial_{+q} := \frac{\partial}{\partial \theta_q^+}. \quad (4.3)$$

These variables have straightforward counterparts in the covariant light-cone basis,  $\theta_q^+ = \theta^\alpha v_{\alpha q}^+$  and  $x^{++} = x^m u_m^{++}$  (see [28, 32] and Sec. 5). The other ‘physical’ variables, on which the wavefunctions should also depend, can be related to other coordinates of this basis, including  $x^{--} = x^m u_m^{--}$  and  $\theta_q^- = \theta^\alpha v_{\alpha q}^-$  and the harmonics  $v_{\alpha q}^-$  parametrizing  $S^9$  (and carrying 9 of 10 degrees of freedom of the light-like momentum). However, to study the cohomology of the BRST operator (4.1) the dependence on these latter coordinates is inessential and, in this section, we will use the notation  $\Phi = \Phi(\lambda_q^+, c^{++}; x^{++}, \theta_q^+, \dots)$  or  $\Phi(c^{++}, \lambda_q^+ \dots)$  to emphasize the essential dependence of our wavefunctions.

The Grassmann odd  $c^{++}$  variable,  $c^{++}c^{++} = 0$ , and the bosonic variables  $\lambda_q^+$  in (4.1) are ghosts for the bosonic and 16 fermionic first class constraints represented by the differential operators  $\partial_{++}$  and  $D_q^-$ . Their ghost numbers are 1, and this also fixes the ghost number of the BRST charge to be one,

$$gh_{\#}(\lambda_q^+) = 1, \quad gh_{\#}(c^{++}) = 1, \quad gh_{\#}(\mathbb{Q}^{susy}) = 1. \quad (4.4)$$

The cohomology problem has to be solved for functions with definite ghost number  $g := gh_{\#}(\Phi)$ . Let us begin, however, with some general observations for which the ghost number fixing is not relevant.

#### 4.1.1 The nontrivial cohomology of $\mathbb{Q}^{susy}$ is located at $\lambda_q^+ \lambda_q^+ = 0$

BRST cohomology is determined by wavefunctions  $\Phi$  which are BRST-closed,  $\mathbb{Q}^{susy}\Phi = 0$ , but not BRST-exact. They are defined modulo the BRST transformations *i.e.* modulo BRST-exact wavefunctions  $\mathbb{Q}^{susy}\chi$ , where  $\chi$  is an arbitrary function of the same configuration space variables and of ghost number  $gh_{\#}(\chi) = gh_{\#}(\Phi) - 1$ ,

$$\mathbb{Q}^{susy}\Phi = 0, \quad \Phi \sim \Phi' = \Phi + \mathbb{Q}^{susy}\chi, \quad gh_{\#}(\chi) = gh_{\#}(\Phi) - 1. \quad (4.5)$$

Decomposing the wave function  $\Phi = \Phi(c^{++}, \lambda_q^+; x^{++}, \theta_q^+, \dots)$  in power series of the Grassmann odd ghost  $c^{++}$ ,

$$\begin{aligned}\Phi &= \Phi_0 + c^{++}\Phi_{++} \\ &= \Phi_0(\lambda_q^+; x^{++}, \theta_q^+; \dots) + c^{++}\Phi_{++}(\lambda_q^+; x^{++}, \theta_q^+; \dots),\end{aligned}\quad (4.6)$$

one finds that  $\mathbb{Q}^{susy}\Phi = 0$  for the superfield (4.6) implies for its components

$$\lambda_q^+ D_q^- \Phi_0 = \lambda_q^+ \lambda_q^+ \Psi_{++} \quad (a), \quad \lambda_q^+ D_q^- \Psi_{++} = i\partial_{++}\Phi_0 \quad (b). \quad (4.7)$$

Using a similar decomposition for the arbitrary superfield in (4.5),  $\chi = \chi_0 + c^{++}K_{++}$ , one finds for the BRST transformations,

$$\Phi \mapsto \Phi' = \Phi + \mathbb{Q}^{susy}\chi \Rightarrow \begin{cases} \Phi_0 \mapsto \Phi'_0 = \Phi_0 + \lambda_q^+ D_q^- \chi_0 - \lambda_q^+ \lambda_q^+ K_{++} & (a), \\ \Psi_{++} \mapsto \Psi'_{++} = \Psi_{++} + i\partial_{++}\chi_0 + \lambda_q^+ D_q^- K_{++} & (b) \end{cases}. \quad (4.8)$$

If one assumes that the spinorial bosonic ghost  $\lambda_q^+$  is non-zero, or, equivalently, that the square  $\lambda_q^+ \lambda_q^+ \neq 0$ , then one can use Eq. (4.7a) to express the fermionic component of the wave function in terms of the bosonic one,  $\Psi_{++} = \lambda_q^+ D_q^- \Phi_0 / \lambda_q^+ \lambda_q^+$ . Then one can also chose the second bosonic component  $K_{++}$  of the parameter superfield  $\chi = \chi_0 + c^{++}K_{++}$  to be  $K_{++} = \frac{1}{\lambda_q^+ \lambda_q^+}(\Phi_0 + \lambda_q^+ D_q^- \chi_0)$  and arrive at  $\Phi'_0 = 0$  in (4.8a). Thus, if the ghost variables  $\lambda_q^+$  parametrize  $\mathbb{R}^{16} - \{0\}$ ,  $\lambda_q^+ \lambda_q^+ \neq 0$  and the BRST cohomology of  $\mathbb{Q}^{susy}$  is necessarily trivial: all the BRST-closed states are BRST-exact.

Hence, if  $\mathbb{Q}^{susy}$  has to admit non-trivial closed states, they must have a representation by wavefunctions with support on  $\lambda_q^+ \lambda_q^+ \neq 0$ . In other words, the closed non-exact wavefunctions representing non-trivial cohomology must be of the form  $\Phi \propto \delta(\lambda_q^+ \lambda_q^+)$  plus a possible BRST trivial contribution.

## 4.2 Cohomologies at vanishing bosonic ghost

Thus wavefunctions describing the non-trivial cohomology of  $\mathbb{Q}^{susy}$ , if exists, must have representation by closed non-exact wavefunctions of the form  $\Phi = \delta(\lambda_q^+ \lambda_q^+) \Phi^{++}$ , where  $\Phi^{++} = \Phi^{++} + c^{++}\Psi^0$  has ghost number two units more than  $\Phi$ ,  $gh_{\#}(\Phi^{++}) = gh_{\#}(\Phi^0) + 2$ . But there is a difficulty with studying these states: since the bosonic ghosts  $\lambda_q^+$  are real,  $\lambda_q^+ \lambda_q^+ = 0$  implies  $\lambda_q^+ = 0$ . Thus, since  $\mathbb{Q}^{susy}$  includes  $\lambda_q^+$  in an essential manner, it is necessary to make a ‘regularization’ allowing us to consider, at the intermediate stages, a nonvanishing  $\lambda_q^+$  which nevertheless satisfies  $\lambda_q^+ \lambda_q^+ = 0$ .

This is possible if we allow  $\lambda_q^+$  to be complex (cf. with the pure spinors by Berkovits [10]),

$$\lambda_q^+ \mapsto \tilde{\lambda}_q^+ : \quad \boxed{\tilde{\lambda}_q^+ \tilde{\lambda}_q^+ = 0, \quad (\tilde{\lambda}_q^+)^* \neq \tilde{\lambda}_q \quad \Rightarrow \quad \tilde{\lambda}_q^+ \neq 0 \text{ is possible}}. \quad (4.9)$$

A suggestive form of the general solution of  $\tilde{\lambda}_q^+ \tilde{\lambda}_q^+ = 0$  is

$$\tilde{\lambda}_q^+ = \epsilon^+ (n_q + im_q), \quad \vec{n}^2 := n_q n_q = 1, \quad \vec{m}^2 := m_q m_q = 1, \quad \vec{n}\vec{m} = n_q m_q = 0, \quad (4.10)$$

where  $n_q$  and  $m_q$  are two real mutually orthogonal unit  $SO(16)$  vectors ( $SO(9)$  spinors) and  $\epsilon^+$  is a real number. The only real representative of the family of complex  $SO(9)$  spinors  $\tilde{\lambda}_q^+$  in (4.10) is  $\tilde{\lambda}_q^+ = 0$ ; this corresponds to setting the ‘regularization parameter’  $\epsilon^+$  equal to zero.

The ‘regularized’ BRST charge is thus complex. It contains the complex ghost  $\tilde{\lambda}_q^+$  rather than the real  $\lambda_q^+$  in (4.1), but does not contain  $(\tilde{\lambda}_q^+)^*$ . It acts on the space of wavefunctions depending, among other configuration space variables, on the complex  $\tilde{\lambda}_q^+$ . Since the discussion

of the previous section is not affected by above complexification  $\lambda_q^+ \mapsto \tilde{\lambda}_q^+$ , we conclude that the non-trivial cohomology states of the complexified BRST charge are wavefunctions of the form

$$\Phi = \delta(\tilde{\lambda}_q^+ \tilde{\lambda}_q^+) \Phi^{++}(\tilde{\lambda}_q^+, c^{++}; x^{++}, \theta_q^+, \dots). \quad (4.11)$$

As the BRST charge  $\mathbb{Q}^{susy}$  does not contain any derivative with respect to the bosonic ghost  $\lambda_q^+$ , its regularization acts on the  $\Phi^{++}$  part of the function  $\Phi$  in (4.11) only. Namely, one finds

$$\mathbb{Q}^{susy}|_{\lambda_q^+ \mapsto \tilde{\lambda}_q^+} \delta(\tilde{\lambda}_q^+ \tilde{\lambda}_q^+) \Phi^{++}(\tilde{\lambda}_q^+, c^{++}; \dots) = \delta(\tilde{\lambda}_q^+ \tilde{\lambda}_q^+) \tilde{Q}^{susy} \Phi^{++}(\tilde{\lambda}_q^+, c^{++}; \dots), \quad (4.12)$$

where we introduced the non-Hermitian BRST charge (*cf.* (4.1))

$$\boxed{\tilde{Q}^{susy} = \tilde{\lambda}_q^+ D_q^- + i c^{++} \partial_{++} \quad , \quad \tilde{\lambda}_q^+ \tilde{\lambda}_q^+ = 0} \quad , \quad \tilde{Q}^{susy} = \mathbb{Q}^{susy}|_{\lambda_q^+ \mapsto \tilde{\lambda}_q^+ : \tilde{\lambda}_q^+ \tilde{\lambda}_q^+ = 0}, \quad (4.13)$$

which can be used to reformulate the regularized cohomology problem. Note that, once we have concluded that the nontrivial cohomology of  $\mathbb{Q}^{susy}$  is determined by wavefunctions of the form (4.11), we can reduce the nontrivial cohomology search to the set of such functions, restricting as well the arbitrary superfields  $\chi$  of the BRST transformations (4.8) to have the form  $\chi = \delta(\tilde{\lambda}_q^+ \tilde{\lambda}_q^+) \chi^{++}$ .

Then, the regularized cohomology problem for the complexified BRST operator ( $\mathbb{Q}^{susy}$  of (4.1) now depending on the complexified bosonic ghost  $\tilde{\lambda}_q^+$ ), reduces to the search for  $\tilde{\lambda}_q^+ = 0$  ‘value’ of the functions describing non-trivial cohomologies of the  $\tilde{Q}^{susy}$  operator in Eq. (4.13),

$$\tilde{Q}^{susy} \Phi^{++} = 0, \quad \Phi^{++} \sim \Phi^{++'} = \Phi^{++} + \tilde{Q}^{susy} \chi^{++}. \quad (4.14)$$

This problem (4.14) can be reformulated in terms of components  $\Phi_0^{++}$  and  $\Psi^{(0)}$  of the wavefunction superfield  $\Phi^{++} = \Phi_0^{++} + c^{++} \Psi^{(0)}$  giving rise to the following equations

$$\tilde{\lambda}_q^+ D_q^- \Phi_0^{++} = 0, \quad \tilde{\lambda}_q^+ D_q^- \Psi^{(0)} = i \partial_{++} \Phi_0^{++}. \quad (4.15)$$

$$\Phi_0^{++} \sim \Phi_0^{++'} = \Phi_0^{++} + \tilde{\lambda}_q^+ D_q^- \chi_0^{++}, \quad \Psi^{(0)} \sim \Psi^{(0)'} = \Psi^{(0)} + i \partial_{++} \chi_0^{++} + \tilde{\lambda}_q^+ D_q^- K^{(0)}. \quad (4.16)$$

To obtain the cohomology of  $\mathbb{Q}^{susy}$ , we have to set  $\tilde{\lambda}_q^+ = 0$  at the end to remove the regularization; thus we are really interested in the wavefunctions for  $\tilde{\lambda}_q^+ = 0$ :  $\Phi_0^{++}(0) := \Phi_0^{++}|_{\tilde{\lambda}_q^+=0} = \Phi_0^{++}(0, x^{++}, \theta_q^+; \dots)$ ,  $\Psi_0^{(0)}(0) := \Psi_0^{(0)}|_{\tilde{\lambda}_q^+=0} = \Psi_0^{(0)}(0, x^{++}, \theta_q^+; \dots)$ .

Eqs. (4.15), (4.16) show that the ‘superfield’ cohomology problem of Eq. (4.14) includes a (pure-spinor like) cohomology problem for the leading component  $\Phi_0^{++}$  of the  $\Phi^{++}$  superfield,

$$\tilde{\lambda}_q^+ D_q^- \Phi_0^{++} = 0, \quad \Phi_0^{++} \mapsto \Phi_0^{++'} = \Phi_0^{++} + \tilde{\lambda}_q^+ D_q^- \chi_0^{++}. \quad (4.17)$$

Let us recall that we are interested in the cohomology problems for fixed ghost number

$$g = gh_{\#}(\Phi) = g_0 - 2, \quad g_0 := gh_{\#}(\Phi_0^{++}). \quad (4.18)$$

As far as the remaining part of the cohomology problem (4.14) (or (4.15), (4.16)) is concerned,

$$\tilde{\lambda}_q^+ D_q^- \Psi^{(0)} = i \partial_{++} \Phi_0^{++}, \quad \Psi^{(0)} \mapsto \Psi^{(0)'} = \Psi^{(0)} + i \partial_{++} \chi_0^{++} + \tilde{\lambda}_q^+ D_q^- K^{(0)}, \quad (4.19)$$

the presence of the  $i \partial_{++} \chi_0^{++}$  term in the BRST transformations suggests its triviality (which is indeed the case, see below).

Thus we have reduced our cohomology problem for the Lorentz harmonics BRST charge (4.1) to the auxiliary cohomology problem (4.17) for the charge (4.13). Before turning to it, we would like to comment on the relation of our BRST charge (4.1) involving a complex  $SO(9)$  spinor  $\tilde{\lambda}_q^+$ , satisfying  $\tilde{\lambda}_q^+ \tilde{\lambda}_q^+ = 0$ , with the Berkovits BRST charge constructed with the  $D=11$  pure spinors [10].

### 4.3 Relation with the Berkovits's pure spinors

The  $D = 11$  pure spinors of Berkovits obey [10]  $\Lambda \Gamma_a \Lambda = 0$  (1.3) and, in general, carry 46 (23 complex) degrees of freedom. A specific 39 parametric solution  $\tilde{\Lambda}$  can be found using the spinor moving frame approach (see [6, 4]). It is given by <sup>12</sup>

$$\boxed{\tilde{\Lambda}_\alpha = \tilde{\lambda}_q^+ v_{\alpha q}^-}, \quad \{v_{\alpha q}^-\} = \frac{SO(1, 10)}{SO(1, 1) \otimes SO(9) \otimes K_9} = S^9, \quad \tilde{\lambda}_q^+ \tilde{\lambda}_q^+ = 0 \Rightarrow \tilde{\Lambda} \Gamma_a \tilde{\Lambda} = 0. \quad (4.20)$$

Thus, the complex 16 component  $SO(9)$  spinor  $\tilde{\lambda}_q^+ \tilde{\lambda}_q^+ = 0$  with  $\tilde{\lambda}_q^+ \tilde{\lambda}_q^+ = 0$ , carries 30 of the 39 degrees of freedom of the (Berkovits-type) pure spinor (4.20). The remaining 9 degrees of freedom in this pure spinor correspond to the  $S^9$  sphere of the light-like eleven-dimensional momentum modulo its energy.

Furthermore, as far as the  $\kappa$ -symmetry generator  $D_q^-$  is basically  $v_q^{-\alpha} \mathbf{d}_\alpha$ , one finds that Berkovits BRST charge in Eq. (1.1) can be obtained from our (4.13) by replacing the composite pure spinor  $\tilde{\lambda}_q^+ v_q^{-\alpha}$  (4.20) by a generic pure spinor  $\tilde{\Lambda}^\alpha$  and by ignoring the second quite simple  $c^{++}$  term in (4.13). In other words,

$$\tilde{Q}^{susy} = \mathbb{Q}^B|_{\Lambda^\alpha = \tilde{\lambda}_q^+ v_q^{-\alpha}} + i c^{++} \partial_{++}, \quad (4.21)$$

Of course, the generic Berkovits's pure spinor [10] in  $D=11$  carries 46 real degrees of freedom, while the composite pure spinor (4.20) only carries 39. However, it is not obvious that all degrees of freedom in a pure spinor are equally important for the description of superparticle in the Berkovits approach. Notice in particular that only the pure spinor cohomology at vanishing bosonic ghost describe the superparticle, while the complete pure spinor cohomology is much richer and correspond to the spinorial cohomologies of [42].

As far as the generalization for the case of superstring is concerned, it is important to note that in  $D = 10$  dimensional case, which corresponds to the Green-Schwarz superstring, Eq. (4.20) does provide the **general solution** of the pure spinor constraint (1.3). Indeed, in  $D = 10$  this solution carries **16+8-2=22** degrees of freedom, the same number as the generic pure spinor. Thus one may expect that the substitution of the solution (4.20) for pure spinor used to describe superstring in [10] (i.e. replacing the pure spinor approach by a pragmatically designed Lorentz harmonic approach) should not produce any additional anomaly.

Coming back to our M0-brane case, we conclude that a counterpart (4.13) of the Berkovits BRST charge (1.1) appears on the way of regularization ( $\lambda_q^- \mapsto \tilde{\lambda}_q^- \neq (\tilde{\lambda}_q^-)^*$ ) from the directly obtained BRST charge (4.1) when the  $D = 11$  superparticle is quantized in its twistor-like Lorentz harmonics formulation (2.8).

### 4.4 Cohomology of $\tilde{\lambda}_q^+ D_q^-$

The physical spectrum of the model is found by solving the BRST cohomology problem in a sector of Hilbert space with a fixed ghost number. When dealing with the  $\Phi_0^{++}$  part of the wavefunction  $\Phi^{++}$ ,  $\Phi^{++} = \Phi_0^{++} + c^{++} \Psi_0$ , the only remaining carrier of the ghost number is the bosonic ghost  $\tilde{\lambda}_q^+$ . Thus the ghost number  $g_0 := g - 2$  of the wavefunction  $\Phi_0^{++}$  (see (4.18)) coincides with its homogeneity degree in  $\tilde{\lambda}_q^+$ ,

$$\Phi_0^{++}(z \tilde{\lambda}_q^+, \dots) = z^{g_0} \Phi_0^{++}(\tilde{\lambda}_q^+, \dots) \Leftrightarrow gh_\#(\Phi_0^{++}) = g_0. \quad (4.22)$$

We are interested in  $\Phi = \delta(\tilde{\lambda}_q^+ \tilde{\lambda}_q^+) \Phi^{++}(\tilde{\lambda}_q^+, \dots)$ , Eq. (4.11) which, after removing regularization, can be written as  $\Phi = \delta(\lambda_q^+ \lambda_q^+) (\Phi_0^{++}|_{\lambda_q^+=0} + c^{++} \Psi_0|_{\lambda_q^+=0})$ . This means that we are actually interested in the cohomologies of the operator  $\tilde{\lambda}_q^+ D_q^-$  at vanishing bosonic ghost,  $\tilde{\lambda}_q^+ = 0$ .

<sup>12</sup>Indeed, using the constraint (2.9) one finds that  $\tilde{\Lambda} \tilde{\Gamma}_a \tilde{\Lambda} = \tilde{\lambda}_q^+ v_q^- \tilde{\Gamma}_a v_p^- \tilde{\lambda}_p^+ = u_a^{-+} \tilde{\lambda}_q^+ \tilde{\lambda}_q^+ = 0$  since  $\tilde{\lambda}_q^+ \tilde{\lambda}_q^+ = 0$ .

As such, one immediately concludes that we cannot have nontrivial cohomology with  $\Phi_0^{++}$  of ghost number  $g_0 > 0$  since, due to (4.22),  $\Phi_{0g_0>0}^{++}(\lambda_q^+ = 0) = 0$ . Furthermore, the values of the ghost number  $g_0 < 0$  are actually prohibited for  $\Phi^{++} = \Phi_0^{++} + \dots$  in (4.11), because  $\Phi_{0g_0<0}^{++} \xrightarrow{\lambda_q^+ \rightarrow 0} \infty$  and the expression for  $\Phi$  in (4.11) diverges (as  $\delta(\lambda^2) \cdot \infty$ ) and cannot describe a physical state. Thus a non-trivial BRST cohomology for (4.1) may come from the  $\tilde{\lambda}_q^+ D_q^-$  cohomologies in the Hilbert space sector of the ghost number  $g_0 = 0$  *only*. This corresponds to  $g := g_0 - 2 = -2$  for the ghost number of the complexified  $\mathbb{Q}^{susy}$ -closed, non-exact wavefunction  $\Phi$  in Eq. (4.11) (see Eq. (4.18)).

Assuming the wavefunctions  $\Phi_0^{++}$  to be analytic in  $\tilde{\lambda}_q^+$ , one finds that, being homogeneous of degree zero, the wavefunction is actually independent of  $\tilde{\lambda}_q^+$ . Then  $\tilde{\lambda}_q^+ D_q^- \Phi_0^{++} = 0$  actually implies  $D_q^- \Phi_0^{++} = 0$ . As far as the BRST transformations  $\Phi_0^{++} \mapsto \Phi_0^{++'} = \Phi_0^{++} + \tilde{\lambda}_q^+ D_q^- \chi_0^{++}$  of Eq. (4.17) are considered, the above assumptions requires  $\chi_0^{++}$  to be an analytic function of  $\tilde{\lambda}_q^+$  with degree of homogeneity  $-1$ , and such a nonvanishing function does not exist.

Hence the calculation of the reduced BRST cohomology (4.17) ( $(\tilde{\lambda}_q^+ D_q^-)$ -cohomology) in the space of the analytic wavefunctions  $\Phi_0^{++}$  of ghost number zero is reduced to calculating the kernel of the  $\tilde{\lambda}_q^+ D_q^-$  operator which, in the sector of ghost number zero, coincides with the kernel,  $D_q^- \Phi_0^{++} = 0$ , of the  $\kappa$ -symmetry generator  $D_q^-$ ,

$$g_0 := gh_{\#} \Phi_0^{++} = 0, \quad \tilde{\lambda}_q^+ D_q^- \Phi_0^{++} = 0 \quad \Rightarrow \quad D_q^- \Phi_0^{++} = 0. \quad (4.23)$$

With the realization (4.3), this equation implies the vanishing of all the coefficients in the decomposition of  $\Phi_0^{++}$  in the power series on  $\theta_q^+$ , and requires that the leading ( $\theta_q^+$  independent) component does not depend on  $x^{++}$ . In other words the general solution of this equation is a function independent on both  $\theta_q^+$  and  $x^{++}$ ,

$$g_0 := gh_{\#} \Phi_0^{++} = 0, \quad \tilde{\lambda}_q^+ D_q^- \Phi_0^{++} = 0 \quad \Rightarrow \quad \begin{aligned} &\Phi_0^{++} \neq \Phi_0^{++}(x^{++}, \theta_q^+) \\ &\left( \frac{\partial}{\partial x^{++}} \Phi_0^{++} = 0, \quad \frac{\partial}{\partial \theta_q^+} \Phi_0^{++} = 0 \right). \end{aligned} \quad (4.24)$$

The ghost number of the second component  $\Psi_0$  of the wavefunction  $\Phi^{++} = \Phi_0^{++} + c^{++} \Psi_0$  is  $gh_{\#}(\Psi_0) = g_0 - 1$ , so that when  $g_0 = 0$  and the nontrivial cohomologies can be carried by  $\Phi_0^{++}$ ,  $gh_{\#}(\Psi_0) = -1$  which, according to the discussion above, requires  $\Psi_0 = 0$ . On the other side, when  $g_0 = 1$  and the wavefunction  $\Phi_0^{++}$  cannot describe a nontrivial cohomology of  $Q_{susy}$ , one can find a nonzero BRST closed  $\Psi_0$  obeying the first equation in (4.19). However, the second equation in (4.19) allows one to ‘gauge’  $\Psi_0$  away by using the parameter  $\chi_0^{++}$  so that the cohomology problem defined by Eqs. (4.19) has only the trivial solution.

Thus the nontrivial *cohomology of the BRST charge*  $\mathbb{Q}^{susy}$  (4.1) is described by the cohomology of the complex  $\tilde{Q}^{susy}$  (4.13) in the sector of ghost number  $g_0 := gh_{\#}(\Phi^{++}) = 0$  (which corresponds to  $g := gh_{\#}(\Phi) = -2$  for  $\Phi$  in (4.11)), which in turn is described by wavefunctions that depend on the ‘physical variables’ only. This actually reduces the covariant quantization problem to the quantization of the physical degrees of freedom, *i.e.* to a counterpart of the twistor quantization presented in [3].

The fact that the cohomologies of the BRST operator are described by wavefunctions that do not depend on variables on which the constraints  $D_q^-$  and  $\partial_{++}$  act nontrivially ( $x^{++}$  and  $\theta_q^+$  in (4.11)) is related to properties that are specific for the superparticle case, where there exists a coordinate basis in which the action is written in terms of variables invariant under both  $\kappa$ -symmetry (generated by  $D_q^-$  above) and  $b$ -symmetry (generated by  $\partial_{++}$ ). The action in such a coordinate basis will be discussed in the next, concluding Sec. 6. Let us note that the above effect does not happen in the superstring case, and hence in the cohomology problem for the superstring counterpart of the BRST charge (4.1) such a simplification cannot occur.



We have to stress that of all the cohomologies of the complex Berkovits-like BRST charge  $\tilde{\mathbb{Q}}^{susy}$  only their values at vanishing bosonic ghost,  $\tilde{\lambda}_q^- = 0$ , describe the cohomologies of the M0-brane BRST charge  $\mathbb{Q}^{susy}$  and, hence, the superparticle spectrum. The  $\tilde{\mathbb{Q}}^{susy}$  cohomologies for  $\tilde{\lambda}_q^- \neq 0$ , corresponding to the higher ghost numbers, are richer and are related with the spinorial cohomologies of [42].

## 5 M0-brane and its quantization in the covariantized light-cone basis.

The simple structure of the cohomology of the M0-brane BRST charge  $\mathbb{Q}^{susy}$  can be explained by studying the spinor moving frame action (2.8) in different basis of canonical variables, particularly in the *covariantized light-cone basis* [28, 29, 32]. The coordinates of this,  $(x^{\pm\pm}, x^i, \theta_q^\pm)$ , are constructed from the ones of the standard basis of superspace  $Z^M = (x^m, \theta^\alpha)$  and harmonics as (see [32], *cf.* [28])

$$x^{\pm\pm} = x^m u_m^{\pm\pm}, \quad x^i = x^m u_m^i, \quad \theta_q^\pm := \theta^\alpha v_{\alpha q}^\pm. \quad (5.25)$$

The change of variables (5.25) in the superparticle action (2.8) gives

$$S := \int d\tau L = \int_{W^1} \left( \frac{1}{2} \rho^{++} D x^{--} - \frac{1}{2} \rho^{++} \Omega^{--i} \tilde{x}^i - i D \theta_q \theta_q \right), \quad (5.26)$$

where

$$\tilde{x}^i = x^i + i \theta_p^- \gamma_{pq}^i \theta_q^+ := x^i + i \theta^\alpha v_{\alpha p}^- \gamma_{pq}^i v_{\beta q}^+ \theta^\beta, \quad D x^{--} := d x^{--} + 2 \Omega^{(0)} x^{--}, \quad (5.27)$$

$$\theta_q = \sqrt{\rho^{++}} \theta_q^- := \sqrt{\rho^{++}} \theta^\alpha v_{\alpha q}^-, \quad D \theta_q := d \theta_q + \frac{1}{4} \Omega^{ij} \theta_p \gamma_{pq}^{ij}, \quad (5.28)$$

and  $\Omega^{(0)}$ ,  $\Omega^{ij}$  are the  $SO(1,1)$  and  $SO(9)$  Cartan forms, see Eq. (3.16).

Notice that the action (5.26) is given in terms of  $\kappa$ - and  $b$ -invariant variables, so that no further gauge fixing is needed. Indeed, the irreducible  $\kappa$ -symmetry of the action (2.8) is characterized by Eq. (2.10),

$$\delta_\kappa x^m = i \delta_\kappa \theta^\alpha \Gamma_{\alpha\beta}^m \theta^\beta, \quad \delta_\kappa \theta^\alpha = \kappa^{+q} v_{q\alpha}^-, \quad \delta_\kappa v_{\alpha q}^- = 0 = \delta_\kappa u_m^{--}. \quad (5.29)$$

For the fermionic coordinate functions in the covariantized light cone basis one finds that  $\theta_q^+$  is transformed additively by the 16-component  $\kappa$ -symmetry parameter,  $\delta_\kappa \theta_q^+ = \kappa^{+q}$ , while  $\delta_\kappa \theta_q^- = 0$ . Furthermore,  $\delta_\kappa x^{++} = 2i \kappa^{+q} \theta_q^+$ , while  $\delta_\kappa x^{--} = 0$  and, although  $\delta_\kappa x^i = i \kappa^{+q} \gamma_{qp}^i \theta_p^-$ ,  $\tilde{x}^i$  of Eq. (5.27) is  $\kappa$ -invariant,  $\delta_\kappa \tilde{x}^i = 0$ . Thus all the variables entering the action (5.26) are inert under  $\kappa$ -symmetry,

$$\delta_\kappa x^{--} = 0, \quad \delta_\kappa \tilde{x}^i := \delta_\kappa x^i - i \kappa^{+q} \gamma_{qp}^i \theta_p^- = 0, \quad \delta_\kappa \theta_q^- = 0, \quad i_\kappa \Omega^{--i} = 0 \quad (5.30)$$

This completes the proof of that just the change of variable (5.25) in the spinor moving frame action (2.8) results in the functional (5.26) which involves  $\kappa$ -invariant variables only. This phenomenon of an automatic gauge fixing, noticed already in [28], explains the mentioned simple structure of the cohomology of the BRST operator constructed from just the  $\kappa$ - and  $b$ -symmetry generators  $D_q^-$  and  $\partial_{++}$ .

The above ‘automatic’ gauge fixing does not occur in the superstring case and, hence, the cohomology of the corresponding Lorentz harmonics BRST operators are expected to be richer.

## 5.1 On BRST quantization of M0-brane in the covariantized light cone basis

Hence, a difference between the original action of Eq. (2.8) and the action in the covariantized light-cone basis (5.25), Eq. (5.26), is that the latter contains only variables invariant under the  $\kappa$ - and  $b$ -symmetries. Thus changing the basis to (5.25) automatically provides the  $\kappa$ -symmetry and  $b$ -symmetry gauge fixed action (this effect was firstly noticed in [28]). Another difference between the two actions is that the harmonics  $v_{\alpha q^-}$  enter in (5.26) *only* through the Cartan forms  $\Omega^{--j}$ ,  $\Omega^{(0)}$ ,  $\Omega^{ij}$  defined by Eqs. (3.19), (3.16)) and entering the canonical Liouville one form on the  $SO(1, D-1)$  group manifold as defined in Eqs. (3.21), (3.25),

$$\frac{1}{2}\Omega^{(a)(b)}\mathbf{d}_{(a)(b)} := -\frac{1}{2}\Omega^{--i}\mathbf{d}^{++i} - \frac{1}{2}\Omega^{++i}\mathbf{d}^{--i} - \Omega^{(0)}\mathbf{d}^{(0)} + \frac{1}{2}\Omega^{ij}\mathbf{d}^{ij}. \quad (5.31)$$

### 5.1.1 Hamiltonian mechanics in the covariantized light-cone basis

Let us define the canonical momenta in the usual way and the covariant canonical momenta by (3.25) and remove the second class constraints on the harmonics by using the Dirac brackets (3.13) (see Sec. 3.2). Doing the same for the fermionic second class constraints, we identify the 16 Grassmann variables with their momenta,

$$\{\theta_q, \theta_p\}_{DB} = -\frac{i}{2}\delta_{qp}. \quad (5.32)$$

Then the bosonic ‘primary’ constraints implied by the action (5.26) read

$$\mathbf{d}^{(0)} + \rho^{++}x^{--} \approx 0, \quad \mathbf{d}^{ij} + \frac{i}{2}\theta\gamma^{ij}\theta \approx 0, \quad \mathbf{d}^{--i} \approx 0, \quad (5.33)$$

$$\mathbf{d}^{++i} - \rho^{++}\tilde{x}^i \approx 0, \quad \tilde{P}_j \approx 0, \quad (5.34)$$

$$P_{--} - \frac{1}{2}\rho^{++} \approx 0, \quad P_{++}^{(\rho)} \approx 0. \quad (5.35)$$

Clearly, the last two constraints, Eqs. (5.35), provide the resolved pair of the second class constraints, which allows us to remove the  $\rho^{++}$  variable by replacing it by  $2P_{--}$ . The same is true about the pairs of constraints in (5.34), which allows us to remove the orthogonal  $\tilde{x}^i$  coordinates (the non-covariant counterparts of which describe the physical degrees of freedom in the standard light-cone gauge description of the Brink-Schwarz superparticle and Green-Schwarz superstring) by expressing them through the covariant momenta  $\mathbf{d}^{++i}$  for the harmonic variables and the  $P_{--}$  momentum

$$\tilde{x}^i = \frac{\mathbf{d}^{++i}}{2P_{--}}. \quad (5.36)$$

The remaining constraints, Eqs. (5.33),

$$\widetilde{\mathbf{d}^{(0)}} := \mathbf{d}^{(0)} + 2x^{--}P_{--} \approx 0, \quad \widetilde{\mathbf{d}^{ij}} := \mathbf{d}^{ij} + \frac{i}{2}\theta\gamma^{ij}\theta \approx 0, \quad \widetilde{\mathbf{d}^{--i}} := \mathbf{d}^{--i} \approx 0, \quad (5.37)$$

are first class ones. Their Dirac brackets produce the  $(so(1,1) \oplus so(9)) \in K_9$  algebra, which can be obtained from the  $so(1,10)$  of Eq. (3.27) by omitting the relations involving  $\mathbf{d}^{++i}$ ,

$$\begin{aligned} [\widetilde{\mathbf{d}^{ij}}, \widetilde{\mathbf{d}^{kl}}]_{DB} &= 2\widetilde{\mathbf{d}^{k[i}\delta^j]l} - 2\widetilde{\mathbf{d}^{l[i}\delta^j]k}, & [\widetilde{\mathbf{d}^{(0)}}, \widetilde{\mathbf{d}^{ij}}]_{DB} &= 0, \\ [\widetilde{\mathbf{d}^{(0)}}, \widetilde{\mathbf{d}^{--i}}]_{DB} &= -2\widetilde{\mathbf{d}^{--i}}, & [\widetilde{\mathbf{d}^{ij}}, \widetilde{\mathbf{d}^{--k}}]_{DB} &= 2\widetilde{\mathbf{d}^{--[i}\delta^j]k}, \\ [\widetilde{\mathbf{d}^{--i}}, \widetilde{\mathbf{d}^{--j}}]_{DB} &= 0. \end{aligned} \quad (5.38)$$

No ‘W-deformation’ occurs here. Actually this is natural, as the *r.h.s.* in Eq. (3.44) was proportional to the square of the  $\kappa$ -symmetry generator absent in the covariantized light-cone basis.

### 5.1.2 BRST charge for the first class constraints in the covariantized light-cone basis

In the covariantized light-cone basis, where the  $\kappa$ -symmetry and  $b$ -symmetry are automatically gauge fixed, the superparticle quantization might be based on the BRST operator for the algebra (5.38) of the  $SO(1,1) \otimes SO(9) \ltimes K_9$  symmetry, appearing here as the full BRST operator for the gauge symmetries of the M0-brane model,

$$\begin{aligned} \mathbf{Q}^{[SO(1,1) \otimes SO(9)] \ltimes K_9} &= c^{++i} D^{--i} + \frac{1}{2} c^{ij} D^{ij} + c^{(0)} D^{(0)} - \\ &\quad - \frac{1}{2} c^{++i} c^{ij} \frac{\partial}{\partial c^{++j}} + 2c^{(0)} c^{++j} \frac{\partial}{\partial c^{++j}} + c^{ik} c^{jk} \frac{\partial}{\partial c^{ij}}. \end{aligned} \quad (5.39)$$

Here  $D^{(0)}$ ,  $D^{ij}$  and  $D^{--i}$  are harmonic covariant derivatives representing the  $SO(1,1)$ ,  $SO(9)$  and  $K_9$  generators and, thus, obeying the Lie algebra

$$\begin{aligned} [D^{(0)}, D^{--i}] &= -2D^{--i}, \quad [D^{ij}, D^{--k}] = 2D^{--[i} \delta^{j]k}, \\ [D^{ij}, D^{kl}] &= 2D^{k[i} \delta^{j]l} - 2D^{l[i} \delta^{j]k}, \quad [D^{(0)}, D^{ij}] = 0, \quad [D^{--i}, D^{--j}] = 0, \end{aligned} \quad (5.40)$$

$c^{(0)}$ ,  $c^{ij}$  and  $c^{++i}$  are the fermionic ghosts for these symmetries and the derivative with respect to the tensorial ghost is defined by  $\frac{\partial c^{i'j'}}{\partial c^{ij}} = 2\delta_{[i}^{i'} \delta_{j]}^{j'}$ .

## 5.2 Covariant quantization of the physical degrees of freedom and hints of hidden symmetries

Although the quantization of the physical degrees of freedom in the covariantized light cone basis (*cf.* [28], where the vector harmonics were used for the first time in quantization of such a type) is similar to the supertwistor quantization of [3], we briefly discuss it here as it gives hints about possible hidden symmetries of the 11D supergravity (see [3] for the discussion on  $SO(16)$ ).

As the first class constraints (5.37) obey the Dirac bracket algebra (5.38) isomorphic to  $[SO(1,1) \otimes SO(9)] \ltimes K_9$  (no deformation appear), we can, following Dirac [40], just impose their quantum counterparts  $D^{(0)}$ ,  $D^{ij}$  and  $D^{--i}$  (5.40) as differential operator conditions on the wavefunction  $\Phi$ ,

$$D^{(0)}\Phi = 0, \quad D^{ij}\Phi = 0, \quad D^{--i}\Phi = 0. \quad (5.41)$$

In the purely bosonic limit the differential equations (5.41) are imposed on the wavefunction which depends on the spinorial harmonics (which, due to the second class constraints, parametrize the  $Spin(1,10)$  group manifold, see Secs. 3.2-2.4) and  $\rho^{++}$ .<sup>13</sup> Imposing the conditions (5.41) is tantamount to requiring that, as a function of harmonics, the wavefunction is now a function on the  $S^9$  sphere which (in the light of the primary constraint (3.1) generalizing the Cartan–Penrose representation for a light-like vector to  $D=11$ ) can be identified with the space of light-like momentum modulo its scale. This scale of the massless particle momentum, the energy, can be identified then (again in the light of the Cartan–Penrose constraint (3.1)) with the Lagrange multiplier  $\rho^{++}$ .

Then, as the canonical Hamiltonian  $H_0$  corresponding to the action (5.26) is zero,  $H_0 \approx 0$ , one concludes that, in the purely bosonic limit, the wavefunction is just an arbitrary function of the above listed physical bosonic variables, namely

$$\Phi|_{\theta_q=0} = \Phi_0(\mathbb{R}_+ \otimes \mathbb{S}^9), \quad \{(v_{\alpha q}^-, \rho^{++})\} = \mathbb{R}_+ \otimes \mathbb{S}^9 = \{(p_{\underline{m}} : p^2 := p_{\underline{m}} p^{\underline{m}} = 0)\}. \quad (5.42)$$

This result coincides with one obtained in [3] in the framework of supertwistor quantization of the M0-brane model.

<sup>13</sup>Alternatively, one can consider a wavefunction dependent on harmonics and  $x^{--}$ , but for our line of arguments the use of wavefunctions dependent on  $\rho^{++}$  ( $= 2P_{--}$ , see (5.35)) is more convenient.

The complete M0-brane action (5.26) includes also the fermionic contribution  $D\theta_q \theta_q = d\theta_q \theta_q + \Omega_{pq}\theta_{[p} \theta_{q]}$ , where  $\Omega_{pq} = -\Omega_{qp} := \frac{1}{4}\Omega^{ij}\gamma_{pq}^{ij}$  is the  $Spin(9)$  connection. Their presence modifies the  $SO(9)$  generator by the term bilinear in fermions (see Eq. (5.37)), but this does not change the conclusion about the wavefunction dependence on the bosonic configurational space coordinates (which, from the spacetime point of view, happen to parametrize the light-like momentum).

Then, the fermionic variables  $\theta_q$  obey the second class constraints stating that they are momenta for themselves, which can be treated in the strong sense after passing to the Dirac brackets (5.32). In quantum theory the Dirac brackets relation (5.32) give rise to the anti-commutational relation stating that the Grassmann coordinate function of the M0-brane becomes a Clifford algebra valued,

$$\{\hat{\theta}_q, \hat{\theta}_p\} = \frac{1}{2}\delta_{qp}, \quad q = 1, 2, \dots, 16. \quad (5.43)$$

This  $O(16)$  covariant Clifford algebra  $\mathcal{Cl}^{16}$  has a finite dimensional representation by  $256 \otimes 256$  sixteen dimensional gamma matrices

$$\hat{\theta}_q = \frac{1}{2}(\Gamma_q)_{\mathcal{A}\mathcal{B}}, \quad \mathcal{A}, \mathcal{B} = 1, \dots, 256, \quad q = 1, 2, \dots, 16. \quad (5.44)$$

Notice that the  $O(16)$  symmetry of the Clifford algebra  $\mathcal{Cl}^{16}$  is the same  $O(16)$  which we have met in the classical analysis of the spinor moving frame action, sec. 2.3. Indeed, it acts in the same way and on the same indices, as far as  $\theta_q = \sqrt{\rho^{++}} \theta^\alpha v_{\alpha q}^-$ , Eqs. (5.25), (5.28). Thus our spinor moving frame formulation (2.8) makes manifest, already at the classical level, the  $SO(16)$  symmetry playing, as we will see in a moment, an important role in the M0-brane quantization.

But before, let us make the following comments.

Firstly, substituting for  $\theta_q$  its contraction with an  $SO(16)$  matrix,  $\theta_q \mapsto \theta_p S_{pq}$  would produce the covariant derivative with the  $SO(16)$  connection  $\Omega_{pq} \mapsto (dS S^T)_{pq} + \frac{1}{4}\Omega^{ij}(S^T \gamma^{ij} S)_{pq}$ ,

$$\begin{aligned} D(\theta S)_q (\theta S)_q &= \tilde{D}\theta_q \theta_q = d\theta_q \theta_q + \tilde{\Omega}_{pq}\theta_{[p} \theta_{q]}, \quad S S^T = I, \\ \tilde{\Omega}_{pq} &= (dS S^T)_{pq} + \frac{1}{4}\Omega^{ij}(S^T \gamma^{ij} S)_{pq} \equiv (dS S^T)_{pq} + \frac{1}{4}\Omega^{ij}((S^T \gamma^{[i} S)(S^T \gamma^{j]} S))_{pq}. \end{aligned} \quad (5.45)$$

It is not evident that such transformation leave the model invariant. To be convinced that they do (when supplemented by the corresponding transformations of the bosonic variables), one can recall that  $\theta_q = \sqrt{\rho^{++}} \theta^\alpha v_{\alpha q}^-$  (Eq. (5.25)), that the action (5.26) is equivalent to (2.8) (obtained from it just by moving derivatives) and that the change  $v_{\alpha q}^- \mapsto v_{\alpha p}^- S_{pq}$  leaves the action (2.8) unchanged as far as  $S S^T = I$  (*i.e.*  $S \in O(16)$ ).

Secondly, taking into account the results of quantization in the bosonic case, in which the state vector is described by the wavefunction of the the light-like momentum,  $\Phi_0 = \Phi_0(p_{\underline{m}}|_{p^2=0})$ , one might think that the state vector of the supersymmetric particle is described by the *Clifford superfield* [48], *i.e.* by the wavefunction dependent on such a light-like momentum  $p_{\underline{m}}$  and on the Clifford algebra valued  $\hat{\theta}_q$  variable,

$$\begin{aligned} \Phi(p_{\underline{m}}|_{p^2=0}, \hat{\theta}_q) &= \Phi_0(p_{\underline{m}}|_{p^2=0}) + 2\hat{\theta}_q \Psi_q(p_{\underline{m}}|_{p^2=0}) + \dots + \frac{2^n}{n!} \hat{\theta}_{q_1} \dots \hat{\theta}_{q_n} \Phi_{q_n \dots q_1}(p_{\underline{m}}|_{p^2=0}) + \\ &+ \dots + \frac{2^{16}}{16!} \hat{\theta}_{q_1} \dots \hat{\theta}_{q_{16}} \Phi_{q_{16} \dots q_1}(p_{\underline{m}}|_{p^2=0}), \quad \hat{\theta}_q \hat{\theta}_p + \hat{\theta}_q \hat{\theta}_q = \frac{1}{2} \delta_{qp} \mathbb{I}, \end{aligned} \quad (5.46)$$

where the coefficients are antisymmetric on their indices,  $\Phi_{q_n \dots q_1}(p_{\underline{m}}|_{p^2=0}) = \Phi_{[q_n \dots q_1]}(p_{\underline{m}}|_{p^2=0})$ .

However, *such a representation of the  $SO(16)$  symmetry is reducible*. It is reducible also as a representation of the Clifford algebra  $\mathcal{Cl}^{16}$ . To see this, one can use the matrix representation (5.44) substituting the sixteen dimensional gamma-matrices for  $2\hat{\theta}_q$ . Then the (5.46) becomes represented

by the  $256 \times 256$  matrix wavefunction,  $\Phi(p_{\underline{m}}|_{p^2=0}, \hat{\theta}_q) \mapsto \Phi_{\mathcal{A}}^{\mathcal{B}}(p_{\underline{m}}|_{p^2=0})$ ,

$$\begin{aligned} \Phi_{\mathcal{A}}^{\mathcal{B}}(p_{\underline{m}}) &:= \Phi_0(p_{\underline{m}})\delta_{\mathcal{A}}^{\mathcal{B}} + \Psi_q(p_{\underline{m}})\Gamma_q \mathcal{A}^{\mathcal{B}} + \dots + \frac{1}{n!}\Phi_{q_n \dots q_1}(p_{\underline{m}})\Gamma_{q_1 \dots q_n} \mathcal{A}^{\mathcal{B}} + \\ &+ \dots + \frac{1}{16!}\Phi_{q_{16} \dots q_1}(p_{\underline{m}})\Gamma_{q_1 \dots q_{16}} \mathcal{A}^{\mathcal{B}}, \quad p^2 = 0. \end{aligned} \quad (5.47)$$

This is a general  $SO(16)$  *bi*-spinor carrying the  $\mathbf{256} \times \mathbf{256}$  representation which is *reducible* both as the representation of the  $SO(16)$  symmetry and of the Clifford algebra  $\mathcal{Cl}^{16}$ .

The appearance of a reducible representation contradicts to the very spirit of the quantization procedure. The result of quantization of a particle mechanics is assumed to be an elementary particle, the definition of which (see *e.g.* [53]) was formulated involving the requirement to be irreducible representation of Poincaré and other physical symmetry groups. This makes accessible the procedure of projecting out a part of quantum state spectrum in quantization of spinning particle [57] and the famous GSO (Gliozzi–Scherk–Olive) projection in quantization of the RNS string model [23].

Hence, the prescription of an unrestricted Clifford superfield does not work, at least in our  $D=11$  massless superparticle case. A simplest irreducible representation of  $\mathcal{Cl}^{16}$  is the  $SO(16)$  Majorana spinor,  $\mathbf{256}$ , and the choice of the wavefunction  $\Phi_{\mathcal{A}}(p_{\underline{m}}|_{p^2=0})$  gives the linearized supergravity supermultiplet (see [2, 3]).

The physical degrees of freedom of the linearized  $D = 11$  supergravity multiplet are described by symmetric traceless  $SO(9)$  tensor  $h_{IJ} = h_{(IJ)}$ , an antisymmetric third rank  $SO(9)$  tensor  $A_{J_1 J_2 J_3} = A_{[J_1 J_2 J_3]}$  and a  $\gamma$ -traceless fermionic  $SO(9)$  vector-spinor  $\Psi_{Ip}$ . Indeed, the solution of the linearized Einstein, three-form gauge field and the Rarita-Schwinger equations can be written in terms of the above  $h_{IJ}$ ,  $A_{J_1 J_2 J_3}$ ,  $\Psi_{Ip}$  and Lorentz harmonics as (see [32, 3])

$$\left\{ \begin{array}{l} h_{mn}(p) = u_m^I u_n^J h_{IJ}(p), \\ A_{mnp}(p) = u_m^I u_n^J u_p^K A_{IJK}(p), \\ \Psi_{m\alpha}(p) = \Psi_{Ip}(p) u_m^I v_{\alpha q}^- \sqrt{\rho^{++}}, \end{array} \quad \begin{array}{l} p_m = \rho^{++} u_m^{--}, \\ u_m^- \Gamma_{\alpha\beta}^m = 2v_{\alpha q}^- v_{\beta q}^- \end{array} \right\} \Rightarrow p^2 = 0. \quad (5.48)$$

The action of  $\hat{\theta}_q$  on these on-shell fields are defined by (see [2] for the light-cone gauge quantization and [3] for the supertwistor quantization)

$$\begin{aligned} 2\hat{\theta}_q h_{IJ} &= \gamma_{qp}^I \Psi_{Jp} + \gamma_{qp}^J \Psi_{Ip}, \\ 2\hat{\theta}_q A_{IJK} &= \gamma_{qp}^{IJ} \Psi_{Kp} + \gamma_{qp}^{KI} \Psi_{Jp} + \gamma_{qp}^{JK} \Psi_{Ip}, \end{aligned} \quad (5.49)$$

$$2\hat{\theta}_q \Psi_{Ip} = \gamma_{qp}^J h_{IJ} + \frac{1}{3!} \left( \gamma_{qp}^{IJ_1 J_2 J_3} - 6\delta^{I[J_1} \gamma_{qp}^{J_2 J_3]} \right) A_{J_1 J_2 J_3}, \quad (5.50)$$

To see that Eq. (5.49) is nothing but an action of the  $d = 16$  gamma matrix (see (5.44)) on one Majorana spinor of  $SO(16)$ , let us begin by splitting the Majorana spinorial representation of  $SO(16)$  on two Majorana–Weyl (MW) spinor representations,  $\mathbf{256} = \mathbf{128} + \mathbf{\bar{128}}$ ,

$$\Phi_{\mathcal{A}}(p_{\underline{m}}|_{p^2=0}) := \begin{pmatrix} \Phi_A(p_{\underline{m}}|_{p^2=0}) \\ \Psi^{\tilde{A}}(p_{\underline{m}}|_{p^2=0}) \end{pmatrix}. \quad (5.51)$$

The observation is that the balance of the bosonic and fermionic degrees of freedom in  $D = 11$  supergravity multiplet is just  $128 + 128$  and that (*e.g.*) the first,  $\mathbf{128}$ , of the above MW spinor representations can be used to describe the physical degrees of freedom of the bosonic fields of the linearized supergravity supermultiplet, while the second,  $\mathbf{\bar{128}}$ , to describe the physical degrees of freedom of gravitino field

$$\Phi_A = \begin{pmatrix} h_{IJ} \\ A_{IJK} \end{pmatrix}, \quad h_{IJ} = h_{(IJ)}, \quad h_{II} = 0, \quad A_{IJK} = A_{[IJK]},$$

$$A = 1, \dots, 128, \quad I, J, K = 1, \dots, 9 \quad \left( \frac{9 \cdot 10}{2} - 1 + \left\{ \frac{3}{9} \right\} = 44 + 84 = 128 \right), \quad (5.52)$$

$$\begin{aligned} \Psi^{\tilde{A}} &= \sqrt{2} \Psi_{Iq}, \quad \Psi_{Iq} \gamma_{qp}^I = 0, \\ \tilde{A} &= 1, \dots, 128, \quad I = 1, \dots, 9, \quad q = 1, \dots, 16 \quad (9 \cdot 16 - 16 = 128). \end{aligned} \quad (5.53)$$

To resume, the Majorana spinor of  $SO(16)$ , (5.51), can be presented as

$$\Phi_A := \begin{pmatrix} \Phi_A \\ \Psi^{\tilde{A}} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} h_{IJ} \\ A_{IJK} \end{pmatrix} \\ \sqrt{2} \Psi_{Iq} \end{pmatrix}, \quad \begin{cases} h_{IJ} = h_{(IJ)}, \quad h_{II} = 0, \\ A_{IJK} = A_{[IJK]}, \\ \Psi_{Iq} \gamma_{qp}^I = 0. \end{cases} \quad (5.54)$$

Finally, assigning the Grassmann parity 0 and 1 to the first and second Majorana–Weyl components, (5.52) and (5.53), of the (momentum representation) wavefunction (5.54), one arrives at the linearized on shell multiplet of  $D = 11$  supergravity.

With the Weyl representation of the gamma-matrices

$$(\Gamma_q)_{\mathcal{A}}^{\mathcal{B}} = \begin{pmatrix} 0 & \sigma_q^A \tilde{B}_A \\ \tilde{\sigma}_q^{\tilde{B}} & 0 \end{pmatrix} \quad (5.55)$$

$$(\sigma_q \tilde{\sigma}_p + \sigma_p \tilde{\sigma}_q) = \delta_{qp} \mathbb{I}_{128 \times 128}, \quad (\sigma_q \tilde{\sigma}_p)_{AB} = \delta_{AB} + \sigma_{qpAB} \quad (5.56)$$

Eqs. (5.49) and (5.50) can be formulated as an action of the  $d=16$  Pauli matrices on two Majorana–Weyl representations of the  $SO(16)$ ,

$$2\hat{\theta}_q \Phi_A = \sigma_{qA}^{\tilde{B}} \Psi^{\tilde{B}}, \quad 2\hat{\theta}_q \Psi^{\tilde{A}} = \tilde{\sigma}_q^{\tilde{A}} \Phi_B. \quad (5.57)$$

This corresponds to the following representation of the  $d = 16$  Pauli matrices algebra (5.56) in terms of  $d = 6$  Dirac matrices  $\gamma_{qp}^I = \gamma_{(qp)}^I$ :

$$\begin{aligned} \sigma_{qA}^{\tilde{B}} &= \left( \frac{\sqrt{2} \gamma^{(I_1}_{qp} \delta^{I_2)J} - \frac{\sqrt{2}}{9} \delta^{I_1 I_2} \gamma_{qp}^J}{\frac{3}{\sqrt{2}} \gamma^{[I_1 I_2}_{qp} \delta^{I_3]J} - \frac{1}{3\sqrt{2}} (\gamma^{I_1 I_2 I_3} \gamma^J)_{qp}} \right) \equiv \left( \frac{\sqrt{2} \gamma^{(I_1}_{qp} \delta^{I_2)J} - \frac{\sqrt{2}}{9} \delta^{I_1 I_2} \gamma_{qp}^J}{\sqrt{2} \gamma^{[I_1 I_2}_{qp} \delta^{I_3]J} - \frac{1}{3\sqrt{2}} (\gamma^{I_1 I_2 I_3} \gamma^J)_{qp}} \right), \\ \tilde{\sigma}_q^{\tilde{B}} &= \left( \sqrt{2} \delta^{J(I_1} \gamma^{I_2)}_{qp} - \frac{\sqrt{2}}{9} \delta^{I_1 I_2} \gamma_{qp}^J \quad \Big| \quad \frac{1}{3\sqrt{2}} (\gamma^J \gamma^{I_1 I_2 I_3})_{qp} - \frac{3}{\sqrt{2}} \delta^{J[I_1} \gamma^{I_2 I_3]}_{qp} \right). \end{aligned} \quad (5.58)$$

Actually, the above results can be used *to speculate about possible  $E_8$  symmetry of the 11D supergravity*. For the 11D supergravity dimensionally reduced down to  $d=3$  this symmetry was conjectured already in [50] and proved in [51]. Recently the appearance of  $E_8$  symmetry in  $D=11$  supergravity was discussed in [52].

Our line is a bit different and refers on the physical degrees of freedom in the supergravity fields, associated to the irreducible representation of  $SO(D-2) = SO(9)$ , as described above, rather than on the compactification of  $D=11$  supergravity to  $d=3$ .

The generators of  $E_8$  can be split onto the set of the generators of its maximal compact subgroup  $SO(16)$ ,  $J_{qp}$ , 128 generators  $Q^A$  collected in the Majorana–Weyl spinor of  $SO(16)$ , whose commutation relations close on the  $SO(16)$  generator,

$$E_8 : \quad [J_{qp}, J_{q'p'}] = 2\delta_{q'[q} J_{p]p'} - 2\delta_{p'[q} J_{p]q'} , \quad (5.59)$$

$$[J_{qp}, Q_A] = \frac{1}{2} \sigma_{pqAB} Q_B, \quad (5.60)$$

$$[Q_A, Q_B] = \sigma_{pqAB} J_{pq}, \quad (5.61)$$

The Jacobi identities are satisfied due to the sigma-matrix identity  $\sigma^{pq}{}_{(AB}\sigma^{pq}{}_{C)D} = 0$ .

Then, in the superparticle quantization above the linearized supergravity multiplet appears in such a way that all the bosonic fields– or, more precisely, their physical components– are collected in the Majorana–Weyl spinor of  $SO(16)$ . This makes tempting to speculate on the relation of the bosonic field of D=11 supergravity with the  $SO(16)$  spinorial generator  $Q^A$  and, further, with the  $E_8/SO(16)$  coset. Furthermore, this suggests the speculation about possible  $E_8$  symmetry of the uncompactified  $D = 11$  dimensional supergravity (i.e. without dimensional reduction to  $d = 3$ ).

Clearly, the linear approximation, which is seen from the superparticle quantization, do not feel the difference between  $E_8$  and its contraction given by extension of  $SO(16)$  by the mutually commuting spinorial generators (which includes  $[Q_A, Q_B] = 0$  instead of  $[Q_A, Q_B] = \sigma_{AB}^{qp} J_{qp}$  in (5.61)). So, to establish the hypothetic  $E_8$  symmetry of the uncompactified  $D = 11$  supergravity, one should define the  $E_8$  transformations on eleven dimensional vielbein  $e_m{}^a(x)$  and gauge field  $A_{mnk}$ <sup>14</sup> and to show that (at least bosonic) supergravity equations are invariant under such transformations. The experience of the description of the hidden  $SO(16)$  symmetry [9] suggests that this  $E_8$  (if exists) might become manifest in a formalism with broken Lorentz invariance. A new suggestion which brings our study is that, a Lorentz symmetry breaking, which is appropriate to find the hidden  $E_8$  (and also  $SO(16)$ ) symmetry might appear to be  $SO(1, 10) \mapsto SO(1, 1) \otimes SO(9)$  (or  $SO(1, 10) \mapsto [SO(1, 1) \otimes SO(9)] \ltimes K_9$ ) rather than  $SO(1, 10) \mapsto SO(1, 2) \otimes SO(8)$  used in [9] to construct the  $SO(16)$  invariant formulation.

A check of whether the  $D = 11$  supergravity has indeed a hidden  $E_8$  symmetry, even without compactification, or the above described  $SO(16)$  invariance of the linearized supergravity and the coincidence of the number of physical polarizations of the bosonic fields of the linearized supergravity multiplet with the dimension of the  $E_8/SO(16)$  coset is purely occasional is an interesting subject for future study.

## 6 Conclusions and outlook

### 6.1 Conclusions

In this paper we have studied the BRST quantization of the M0-brane in the framework of its spinor moving frame formulation [4, 3] (see [5, 34] for  $D = 4$  and 10) where the action includes the spinorial Lorentz harmonics as twistor-like auxiliary variables. Our main motivation was to search for the origin and geometrical meaning of the properties of the pure spinor approach to the quantum superparticles and superstrings [10].

We have constructed here the Hamiltonian mechanics of the  $D=11$  massless superparticle in the spinor moving frame formulation separating covariantly the first and the second class constraints (which has been possible due to the use of spinorial harmonics [6, 7]) and defining the Dirac brackets allowing to treat the second class constraints as strong equalities.

We have shown that the set of the first class constraints of the M0-brane in the spinor moving frame formulation can be separated into two groups. The first one includes the 16 fermionic generators of the  $\kappa$ -symmetry (which is irreducible in the spinor moving frame formulation due to the presence of spinorial harmonics) and one bosonic generator of the  $b$ -symmetry. These generate the  $d = 1, N = 16$  supersymmetry gauge supergroup  $\Sigma^{(1|16)}$ . The remaining first class constraints correspond to the generators of  $H = [SO(1, 1) \times SO(9)] \ltimes K_9$  gauge symmetry. This eliminates the excess of variables in the harmonics used to formulate the massless  $D=11$  superparticle model making them the homogeneous coordinates of  $S^9$  which can be identified as D=11 celestial sphere.

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<sup>14</sup>The inclusion of fermions is a separate problem; usually, when the  $E_n$  symmetries of the compactified (to  $d = 11 - n$ ) supergravity are considered, the fermions are transformed as the field on nonlinear realization.

However, the superalgebra of the Dirac brackets of the first class constraints is given by a ‘*W*-deformation’ of the one of the semidirect product  $H \ltimes \Sigma^{(1|16)}$ , rather than by this semidirect product itself. This ‘*W*-deformation’ is produced by the appearance of the product of two  $\kappa$ -symmetry generators in the Dirac brackets of two  $K_9$  generators, so that  $K_9$  is no longer an abelian subgroup and the Dirac brackets describes a generalized subalgebra of the enveloping superalgebra rather than a Lie superalgebra.

The structure of the complete BRST charge  $Q$  for all the first class constraints of the M0-brane model is too complicated and its use is not practical. This can be seen already from the BRST charge  $Q'$  for the nonlinear algebra of the  $\kappa$ -symmetry,  $b$ -symmetry and the deformed  $K_9$  symmetry which we have constructed in this paper (3.51). It already contains seven terms with up to fourth power of the ghost fields. In the search for a counterpart of (or even an alternative for) the Berkovits BRST charge we have accepted a further reduction of  $Q'$  down to the simple BRST charge  $Q^{susy}$  (3.52) associated to the  $\kappa$ - and  $b$ -symmetry gauge supergroup  $\Sigma^{(1|16)}$ .

We have shown that the non-trivial cohomologies of  $Q^{susy}$  can be described by wavefunctions which have support on  $\lambda_q^+ \lambda_q^+ = 0$ . This condition requires the bosonic ghost  $\lambda_q^+$ , corresponding to the  $\kappa$ -symmetry, to be zero. Since  $\lambda_q^+$  defines essentially the BRST charge  $Q^{susy}$ , this makes a regularization necessary. Such a regularization is made by allowing the  $\kappa$ -symmetry bosonic ghost to become complex,  $\lambda_q^+ \mapsto \tilde{\lambda}_q^+ \neq (\lambda_q^+)^*$ , and by considering the non-Hermitian BRST charge  $\tilde{Q}^{susy}$  resulting from it. The cohomology of the original BRST charge  $Q^{susy}$  is then given by the cohomology of its complexified and further reduced version  $\tilde{Q}^{susy}$  (Eq. (4.13)) at zero value of the bosonic ghost.

The need for a complex BRST charge at the regularization stage when computing the non-trivial cohomology shows a reason for the intrinsic complexity of the Berkovits pure spinor formalism for the superparticles and the superstring. This conclusion is further supported by the observation that our  $\tilde{Q}^{susy}$  is essentially a particular case of the Berkovits BRST charge for  $D = 11$  superparticle, but with a composite pure spinor constructed from the  $\kappa$ -symmetry ghost and Lorentz harmonics (Eq.(4.20), see also below).

Computing the cohomology of the BRST charge  $Q^{susy}$  we have found that it is nontrivial only in the sector with ghost number  $-2$  (which corresponds to the ghost number  $g_0 = 0$  for the wavefunctions describing cohomologies of  $\tilde{Q}^{susy}$ ) and are essentially described by functions depending only on the physical variables, which are inert under both the fermionic  $\kappa$ - and bosonic  $b$ - gauge symmetries. The reason for such a simple structure is the existence of a specific coordinate basis, the covariantized light-cone basis, the transition to which results in the disappearance from the action of all the worldline fields that transform nontrivially under the  $\kappa$ - and the  $b$ - gauge symmetries.

We have studied the covariant quantization of the physical degrees of freedom in the covariant light-cone basis. This quantization, quite close to the supertwistor one in [3], shows the hints of possible hidden symmetries of D=11 supergravity (or, probably, of M-theory). These include the  $SO(16)$  already mentioned in [3] (and presumably related with the one of [9]), but also some indication of possible  $E_8$ , which brings us quite close to the  $E_{10}$  and  $E_{11}$  busyness of [55] and [56].

## 6.2 Outlook 1: on BRST charge for superstring

The main conclusion of our present study of the M0 case is that the twistor-like Lorentz harmonic approach [6, 3], originated in [28, 29, 30], is able to produce a simple and practical BRST charge. This suggests a similar investigation of the  $D = 10$  Green-Schwarz superstring case. For instance, for the IIB superstring the Berkovits BRST charge looks schematically like

$$Q_{IIB}^B = \int \Lambda^{\alpha 1} d_\alpha + \int \Lambda^{\alpha 2} d_\alpha^2, \quad \Lambda^{\alpha 1} \sigma_{\alpha\beta}^a \Lambda^{\beta 1} = 0 = \Lambda^{\alpha 1} \sigma_{\alpha\beta}^a \Lambda^{\beta 1} \quad (6.1)$$



with two complex pure spinors  $\Lambda^{\alpha 1}$  and  $\Lambda^{\alpha 2}$ . By analogy with our study of M0-brane (see (4.20)), one may expect that the BRST quantization of the of the Green–Schwarz superstring in its spinor moving frame formulation [6, 7] would result, after some reduction and on the way of regularization of the ‘honest’ (‘true’) hermitian BRST charge, in a complex charge of the form (6.1), but with composite pure spinors

$$\tilde{\Lambda}^{\alpha 1} = \tilde{\lambda}_p^+ v_p^{-\alpha}, \quad \tilde{\Lambda}^{\alpha 2} = \tilde{\lambda}_p^- v_p^{+\alpha}, \quad \tilde{\lambda}_p^+ \tilde{\lambda}_p^+ = 0 = \tilde{\lambda}_p^- \tilde{\lambda}_p^- . \quad (6.2)$$

Here, the  $\tilde{\lambda}_p^{\pm}$  are two complex eight component  $SO(8)$  spinors and the stringy harmonics  $v_p^{\mp\alpha}$  are the homogeneous coordinates of the non-compact 16-dimensional coset

$$\{V_{(\beta)}^{\alpha}\} = \{(v_p^{+\alpha}, v_p^{-\alpha})\} = \frac{Spin(1,9)}{SO(1,1) \otimes SO(8)}, \quad (6.3)$$

characteristic for the spinor moving frame formulation of the (super)string [6, 7] and describing the spontaneous breaking of the spacetime Lorentz symmetry by the string model.

It worth noticing that, in contrast with the M0-brane case, the  $D = 10$  solution (6.2) of the pure spinor constraints in (6.1) *carries the same number of degrees of freedom*,  $44 (= 2 \times 8 + 2 \times 14)$ , that the pair of Berkovits complex pure spinors  $\Lambda^{\alpha 1}, \Lambda^{\alpha 2}$  ( $22 + 22$ ). Hence it provides *the general solution* of the  $D = 10$  pure spinor constraints in terms of harmonics (6.3) and two complex  $SO(8)$  spinors of zero square so that its substitution for the generic pure spinor of [10] should not produce any anomaly or other problem related to the counting of degrees of freedom.

### 6.3 Outlook 2: $SO(16)$ , $E_8$ and al that.

Searching for the explanation of simple structure of the cohomologies of the M0-brane BRST charge  $\mathbb{Q}^{susy}$  we studied the M0-brane model in different, the so-called covariantized light cone basis [32], the counterpart of which was first considered in [28]. The change of variables to this basis removes automatically all the worldline fields which transformed nontrivially under the  $\kappa$ -symmetry and  $b$ -symmetry. Such a phenomenon of automatical gauge fixing was first described in [28]; one might observe it as well when passed to the pure (super)twistor form of the action, as in [3].

Quantizing superparticle in this coordinate basis (as well as in the supertwistor one [3]) one easily sees the  $SO(16)$  symmetry of the model<sup>15</sup>. The reason is that, both in the covariantized light cone basis and after fixing the usual light-cone and the (non-covariant)  $\kappa$ -symmetry gauge, the superparticle action contains a set of 16 fermionic fields which, upon quantization, become the  $Cl^{16}$  Clifford algebra valued. The supergravity multiplet appears in the superparticle spectrum when one choose the wavefunction to be in **256** Majorana spinor representation of  $Cl^{16}$ . The bosonic and fermionic fields of the supermultiplet appear as different (**128** and **128**) Majorana Weyl parts of this Majorana spinor.

Furthermore, the observation of the well-known fact that  $E_8$  exceptional group Lie algebra can be written in terms of the generators of  $SO(16)$  and 128 bosonic generators carrying the Majorana spinor (**128**) representation of  $SO(16)$  makes it tempting to speculate on that the  $E_8$  symmetry might be characteristic of the  $D = 11$  supergravity itself rather than of its reduction to  $d = 3$  only. In such a scenario the bosonic fields of the  $D=11$  supergravity multiplet appear to be associated to the generators of the  $E_8/SO(16)$  coset. Notice that the assumption on the Goldstone nature of graviton (physical degrees of freedom in our case) is very much in spirit of the  $E_{11}$  activity of [56], which develops in this respect the line of Borisov and Ogievetsky [58]. Also similarly to the case of  $E_{10}$  and  $E_{11}$  conjecture(s), the fermionic field (gravitino) appears to be out of the consideration and have to be considered as a ‘field of nonlinear realization’ [59].

<sup>15</sup>In our spinor moving frame or twistor-like Lorenz harmonics formulation [4, 34, 3, 41] this symmetry can be seen also at the classical level (see sec. 2.3); in the standard Brink–Schwarz formulation it is hidden and appears after quantization in the light-cone gauge.

Surely, the superparticle quantization provides us only with the linearized fields describing on-shell degrees of freedom. A check of whether the  $D = 11$  supergravity has indeed a hidden  $E_8$  symmetry, even without compactification, or the above described  $SO(16)$  invariance of the linearized supergravity and the coincidence of the number of physical polarizations of the bosonic fields of the linearized supergravity multiplet with the dimension of the  $E_8/SO(16)$  coset is purely occasional, is an interesting subject for future study.

Let us notice that  $E_n/H_n$  cosets, which appeared as a manifold of scalar fields for the  $d = 11 - n$  compactifications of D=11 supergravity, were considered recently in [60] in relation with the M-theoretic generalizations of the Hitchin's generalized geometries [61]. In particular, it was shown [60], that  $E_7/SU(8)$  and  $E_6/Sp(4)$  cosets can be described by  $n$ -dimensional components of the bosonic fields of supergravity,  $g_{ij}$ ,  $A_3$  and  $A_6$  (metric, three form gauge field, and its 11-dimensional dual). The  $E_n/H_n$  cosets with  $n < 6$  can be described by the  $n$ -dimensional components of  $g_{ij}$  and  $A_3$ . The 128-dimensional  $n = 8$  coset  $E_8/SO(16)$  does not feet in this picture. Indeed, it is easy to see that the number of the components of 8-dimensional  $g_{ij}$ ,  $A_3$  and  $A_6$ , is  $36 + 56 + 28 = 120 < 128$ <sup>16</sup>. In the light of this the coincidence of the number of parameter of the  $E_8/SO(16)$  coset with the number of polarizations of the physical bosonic fields of the supergravity multiplet, observed in sec. 5.2 and discussed above, looks even more intriguing and worth further thinking.

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**Notice added in proofs.** When the present work was finished, the author became aware of the work [62] in which the possible hidden  $E_8 \times SO(16)$  symmetry of D=11 supergravity was conjectured for the first time.

### APPENDIX. Derivative of Harmonic variables and $SO(1, 10)$ Cartan forms

Vector ( $u$ ) and spinor ( $v$ ) Lorentz harmonics are elements of  $SO(1, 10)$  and  $Spin(1, 10)$  matrices,  $U \in SO(1, 10)$ ,  $V \in Spin(1, 10)$ , Eqs. (2.17), (2.20). Their interrelation is described by the constraints (2.21) which can be specified as

$$\begin{aligned} 2v_{\alpha q}^- v_{\beta q}^- &= \Gamma_{\alpha\beta}^m u_m^{--} & (a) , & & v_q^- \tilde{\Gamma}_m v_p^- &= u_m^{--} \delta_{qp} & (d) , \\ 2v_{\alpha q}^+ v_{\beta q}^+ &= \Gamma_{\alpha\beta}^m u_m^{++} & (b) , & & v_q^+ \tilde{\Gamma}_m v_p^+ &= u_m^{++} \delta_{qp} & (e) , \\ 2v_{(\alpha|q}^+ \gamma_{qp}^i v_{|\beta)q}^+ &= \Gamma_{\alpha\beta}^m u_m^i & (c) , & & v_q^- \tilde{\Gamma}_m v_p^+ &= u_m^i \gamma_{qp}^i & (f) , \end{aligned} \quad (A.1)$$

The tangent space to the group can be associated to its Lie algebra. A basis of the 55 dimensional  $so(1, 10)$  algebra is provided by antisymmetric tensor generator  $\mathbb{T}_{(a)(b)} = -\mathbb{T}_{(b)(a)}$ ,  $(a) = (++, --, i)$ . The dual space is spanned by the 55 left-invariant Cartan forms  $\Omega^{(a)(b)} = -\Omega^{(b)(a)}$  on the  $SO(1, 10)$  group manifold. This can be expressed in terms of vector harmonics,

$$\Omega^{(a)(b)} := U^{m(a)} dU_m^{(b)} = -\Omega^{(b)(a)} = \begin{pmatrix} 0 & -4\Omega^{(0)} & \Omega^{++j} \\ 4\Omega^{(0)} & 0 & \Omega^{--j} \\ -\Omega^{++i} & -\Omega^{--i} & \Omega^{ij} \end{pmatrix} . \quad (A.2)$$

<sup>16</sup>Of course, the simplest proposition to feet the coset dimension (128) would be to add the eight-dimensional one form  $A_1$ , but this field, in contrast with  $g_{ij}$ ,  $A_3$  and  $A_6$ , does not have a straightforward D=11 origin.

Eq. (A.2) provides a vector representation for the  $so(1,10)$  valued canonical one-form  $g^{-1}dg = \frac{1}{2}\Omega^{(a)(b)}\mathbb{T}_{(a)(b)}$ : with  $\mathbb{T}_{(a)(b)(c)}^{(d)} = 2\eta_{(c)[(a)}\delta_{(b)]}^{(d)}$  and  $g$  expressed in terms of the vector Lorentz harmonics (2.17),  $g = U$ ,  $g^{-1}dg = U^{-1}dU$ .

The covariant harmonic derivatives  $\mathbb{D}_{(a)(b)}$  provides a realization of the generators  $\mathbb{T}_{(a)(b)}$  in terms of the differential operators (vector fields) on the Lorentz group manifold. They can be obtained by decomposing the exterior derivative  $d$  in the basis of the Cartan forms (A.2),

$$d := \frac{1}{2}\Omega^{(a)(b)}\mathbb{D}_{(a)(b)} =: \Omega^{(0)}\mathbb{D}^{(0)} + \frac{1}{2}\Omega^{++i}\mathbb{D}^{--i} + \frac{1}{2}\Omega^{--i}\mathbb{D}^{++i} - \frac{1}{2}\Omega^{ij}\mathbb{D}^{ij}. \quad (\text{A.3})$$

For the light-like vector  $u_m^{--}$ , which is included in the action (2.7), Eqs. (A.3) and (A.2) imply

$$du_m^{--} = -2u_m^{--}\Omega^{(0)} + u_m^i\Omega^{--i}. \quad (\text{A.4})$$

The presence of  $u_m^i$  in the *r.h.s.* of this equation shows the convenience of treating the light-like vector  $u_m^{--}$  in (2.7) as an element of the moving frame: its derivatives (or variations) are given in terms of variables already in the theory.

Notice that the Cartan forms  $\Omega^{(0)}$  and  $\Omega^{ij}$  transform as (composite) gauge fields under the local  $SO(1,1)$  and  $SO(9)$  transformations, respectively. This allows to introduce the  $SO(1,1) \otimes SO(9)$  covariant differential  $D$  and to write Eq. (A.4) and the expressions for the derivatives of other vector harmonics in the form of

$$Du_m^{--} := du_m^{--} + 2u_m^{--}\Omega^{(0)} = u_m^i\Omega^{--i}, \quad (\text{A.5})$$

$$Du_m^{++} := du_m^{++} - 2u_m^{++}\Omega^{(0)} = u_m^i\Omega^{++i}, \quad (\text{A.6})$$

$$Du_m^i := du_m^i + u_m^j\Omega^{ji} = \frac{1}{2}u_m^{++}\Omega^{--i} + \frac{1}{2}u_m^{--}\Omega^{++i}, \quad (\text{A.7})$$

When  $g$  is realized in terms of the spinorial harmonics matrix  $V$ ,  $g^{-1}dg = V^{-1}dV = \frac{1}{2}\Omega^{(a)(b)}\mathbb{T}_{(a)(b)}$ , where now the Lorentz algebra generators are in the spinorial representation,  $\mathbb{T}_{(a)(b)} = \frac{1}{2}\Gamma_{(a)(b)} \in \text{spin}(1,10)$ ,

$$V^{-1}dV = \frac{1}{4}\Omega^{(a)(b)}\Gamma_{(a)(b)} \in \text{spin}(1,10), \quad \Omega^{(a)(b)} := U^{m(a)}dU_m^{(b)} \in so(1,10). \quad (\text{A.8})$$

Eq. (3.18) can be equivalently written in the form of  $dV = \frac{1}{4}\Omega^{(a)(b)}V\Gamma_{(a)(b)}$ . This equation implies, in particular, the following expression for the differential  $dv_{\alpha q}^-$  of the harmonics  $v_{\alpha q}^-$ :

$$dv_q^- = -\Omega^{(0)}v_q^- - \frac{1}{4}\Omega^{ij}v_p^- \gamma_{pq}^{ij} + \frac{1}{2}\Omega^{--i}\gamma_{qp}^i v_p^+ \quad (\text{A.9})$$

In terms of  $SO(1,1) \otimes SO(9)$  covariant derivative this equation and its companion read

$$Dv_q^- := dv_q^- + \Omega^{(0)}v_q^- + \frac{1}{4}\Omega^{ij}v_p^- \gamma_{pq}^{ij} = \frac{1}{2}\Omega^{--i}\gamma_{qp}^i v_p^+, \quad (\text{A.10})$$

$$Dv_q^+ := dv_q^+ - \Omega^{(0)}v_q^+ + \frac{1}{4}\Omega^{ij}v_p^+ \gamma_{pq}^{ij} = \frac{1}{2}\Omega^{++i}\gamma_{qp}^i v_p^-. \quad (\text{A.11})$$

The covariant derivatives of the inverse harmonics, which in our  $D = 11$  case are related to the original ones by

$$D = 11 : \quad v_q^{\pm\alpha} = \pm i C^{\alpha\beta} v_{\beta q}^{\pm} \quad (\text{A.12})$$

have the related but not identical form

$$Dv_q^{-\alpha} := dv_q^{-\alpha} + \Omega^{(0)}v_q^{-\alpha} + \frac{1}{4}\Omega^{ij}v_p^{-\alpha} \gamma_{pq}^{ij} = -\frac{1}{2}\Omega^{--i}v_p^{+\alpha} \gamma_{pq}^i, \quad (\text{A.13})$$

$$Dv_q^{+\alpha} := dv_q^{+\alpha} - \Omega^{(0)}v_q^{+\alpha} + \frac{1}{4}\Omega^{ij}v_p^{+\alpha} \gamma_{pq}^{ij} = -\frac{1}{2}\Omega^{++i}v_p^{-\alpha} \gamma_{pq}^i. \quad (\text{A.14})$$

The minus sign in the *r.h.s.* of (A.13) guaranties that, *e.g.*  $Dv_q^{-\alpha} v_{\alpha p}^+ = -v_q^{-\alpha} Dv_{\alpha p}^+$ ,

$$v_q^{-\alpha} Dv_{\alpha p}^- = -Dv_q^{-\alpha} v_{\alpha p}^- = \frac{1}{2} \Omega^{- - i} \gamma_{pq}^i, \quad v_q^{+\alpha} Dv_{\alpha p}^+ = -Dv_q^{+\alpha} v_{\alpha p}^+ = \frac{1}{2} \Omega^{+ + i} \gamma_{pq}^i. \quad (\text{A.15})$$

Actually, the above equations can also be written with noncovariant derivatives,

$$dv_q^{-\alpha} v_{\alpha p}^- = -v_q^{-\alpha} dv_{\alpha p}^- = -\frac{1}{2} \Omega^{- - i} \gamma_{pq}^i, \quad dv_q^{+\alpha} v_{\alpha p}^+ = -v_q^{+\alpha} dv_{\alpha p}^+ = -\frac{1}{2} \Omega^{+ + i} \gamma_{pq}^i. \quad (\text{A.16})$$

The fact that Cartan forms  $\Omega^{(0)}$  and  $\Omega^{ij}$  are used as  $SO(1,1)$  and  $SO(9)$  connection used to define covariant derivative (A.13), (A.14) can be expressed by

$$v_q^{-\alpha} Dv_{\alpha p}^+ = 0, \quad v_q^{+\alpha} Dv_{\alpha p}^- = 0. \quad (\text{A.17})$$

The Cartan forms  $\Omega^{++i}$  and  $\Omega^{--i}$  are covariant with respect to  $SO(1,1) \otimes SO(9)$  transformations. They provide the vielbein for the coset  $\frac{SO(1,10)}{[SO(1,1) \otimes SO(9)]}$  characteristic of the string rather than of a particle model (*cf.* (2.23)), see Eq. (6.3) and [6, 7] for the  $D = 10$  counterpart. Under the  $K_9$  transformations (2.24), which act on the vector harmonics by

$$K_9 : \quad \delta u_m^{--} = 0, \quad \delta u_m^{++} = 2k^{++i} u_m^i, \quad \delta u_m^i = k^{++i} u_m^{--}, \quad (\text{A.18})$$

the Cartan forms  $\Omega^{++i}$  transform as a connection,  $\delta \Omega^{++i} = 2Dk^{++i} := 2(dk^{++i} + k^{++j} \Omega^{ji} - 2k^{++i} \Omega^{(0)})$ , while  $\Omega^{--i}$  is invariant. This indicates that the Cartan form  $\Omega^{--i}$  provide the vielbein for the coset  $SO(1,10)/[(SO(1,1) \otimes SO(9))] \ltimes K_9 = S^9$ .

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